Port-Hamiltonian systems on multidimensional spatial domains*

ConFlex Project Meeting, August 5th, 2021

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*This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 765579.

Outline of the Talk

- Well-posedness of the inner dynamic

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- Stability of the wave equation

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- Stability of the wave equation
- Compact embeddings

System equation

Let $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary

$$\frac{\partial}{\partial t} x(t,\xi) = \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} \begin{bmatrix} 0 & L_i \\ L_i^{\mathsf{T}} & 0 \end{bmatrix} \mathcal{H}(\xi) x(t,\xi) + P_0 \mathcal{H}(\xi) x(t,\xi), \qquad t \in \mathbb{R}_+, \xi \in \Omega,$$

$$x(0,\xi) = x_0(\xi), \qquad \xi \in \Omega.$$

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Question

What boundary conditions make this system well-posed (existence and uniqueness of solutions)?

Notation

Let $L = (L_i)_{i=1}^n$, where $L_i \in \mathbb{R}^{m_1 \times m_2}$ and ν the normal vector on $\partial\Omega$. We define

$$L_\partial := \sum_{i=1}^n \frac{\partial}{\partial \xi_i} L_i \quad \text{and} \quad L_\nu := \sum_{i=1}^n \nu_i L_i.$$

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The corresponding domain is

$$\mathsf{H}(L_\partial, \Omega) := \{ f \in \mathsf{L}^2(\Omega) \mid L_\partial f \in \mathsf{L}^2(\Omega) \}.$$

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This means we can rewrite the PDE

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if we set $x(t) = x(t,\cdot)$ as

$$\dot{x}(t) = \begin{bmatrix} 0 & L_\partial \\ L_\partial^\mathsf{T} & 0 \end{bmatrix} \mathcal{H}x(t) + P_0\mathcal{H}x(t).$$

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We regard this abstract ODE on $\mathcal{X}_\mathcal{H} = \mathsf{L}^2(\Omega)$ equipped with

$$\langle x, y\rangle_{\mathcal{X}_\mathcal{H}} = \frac{1}{2}\langle x, \mathcal{H}y\rangle_{\mathsf{L}^2(\Omega)}.$$

The energy of a state $x \in \mathcal{X}_\mathcal{H}$ is

$$E(x) = \|x\|_{\mathcal{X}_\mathcal{H}}^2 = \langle x, x\rangle_{\mathcal{X}_\mathcal{H}}$$

Energy

The change of energy along solutions is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(x(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), x(t)\rangle_{\mathcal{X}_{\mathcal{H}}} = \langle \dot{x}(t), x(t)\rangle_{\mathcal{X}_{\mathcal{H}}} + \langle x(t), \dot{x}(t)\rangle_{\mathcal{X}_{\mathcal{H}}} \\ &= \mathrm{Re}\langle A\mathcal{H}x(t), x(t)\rangle_{\mathcal{X}_{\mathcal{H}}}\end{aligned}$$

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$$= 2\,\mathrm{Re}\langle A\mathcal{H}x(t), \mathcal{H}x(t)\rangle_{\mathsf{L}^2(\Omega)} \leq 0 \quad \text{(dissipative)}$$

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The original question can be reduced to

Question

Which boundary condition make A a generator of a contraction semigroup?

Example

$$\operatorname{rot} E = \begin{bmatrix} \partial_2 E_3 - \partial_3 E_2 \\ \partial_3 E_1 - \partial_1 E_3 \\ \partial_1 E_2 - \partial_2 E_1 \end{bmatrix} = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

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$$= \left(\partial_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \partial_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \partial_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

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Hence, $L_\partial = \operatorname{rot}$.

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Hence, $L_\partial = \operatorname{rot}$.

$$L_\nu E = \begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \nu \times E$$

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Note $L_i = -L_i^\top$, consequently $L_\partial^\top = -L_\partial = -\operatorname{rot}$.

$$\begin{bmatrix} \dot{D} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & L_\partial \\ L_\partial^\top & 0 \end{bmatrix} \begin{bmatrix} D \\ B \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{bmatrix} \begin{bmatrix} D \\ B \end{bmatrix}$$

Boundary triples

Let $\mathcal{B}$ a Hilbert space such that $\Gamma_1, \Gamma_2 \colon \operatorname{dom} A \to \mathcal{B}$ is continuous.
$(\mathcal{B}, \Gamma_1, \Gamma_2)$ is a boundary triple for A, if

- $$\langle Ax, y\rangle + \langle x, Ay\rangle = \langle \Gamma_1 x, \Gamma_2 y\rangle + \langle \Gamma_2 x, \Gamma_1 y\rangle$$
- and $\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \colon \operatorname{dom} A \to \mathcal{B} \times \mathcal{B}$ is surjective.

Boundary operators

By the definition of L_∂ and $L_\partial^\top$ we have for $x, y \in \mathsf{H}^1(\Omega)$

$$\langle L_\partial x, y \rangle_{\mathsf{L}^2(\Omega)} + \langle x, L_\partial^\top y \rangle_{\mathsf{L}^2(\Omega)} = \int_\Omega \sum_{i=1}^n \langle \partial_i L_i x, y \rangle + \langle x, \partial_i L_i^\top y \rangle \,\mathrm{d}\lambda$$

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&= \int_\Omega \sum_{i=1}^n \langle \partial_i L_i x, y \rangle + \langle L_i x, \partial_i y \rangle \,\mathrm{d}\lambda \\
&= \int_\Omega \sum_{i=1}^n \partial_i \langle L_i x, y \rangle \,\mathrm{d}\lambda \\
&= \int_{\partial\Omega} \sum_{i=1}^n \nu_i \langle L_i x, y \rangle \,\mathrm{d}\mu
\end{aligned}$$

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$$\begin{aligned}
\langle L_\partial x, y\rangle_{\mathsf{L}^2(\Omega)} + \langle x, L_\partial^\mathsf{T} y\rangle_{\mathsf{L}^2(\Omega)} &= \int_\Omega \sum_{i=1}^n \langle \partial_i L_i x, y\rangle + \langle x, \partial_i L_i^\mathsf{T} y\rangle \,\mathrm{d}\lambda \\
&= \int_\Omega \sum_{i=1}^n \langle \partial_i L_i x, y\rangle + \langle L_i x, \partial_i y\rangle \,\mathrm{d}\lambda \\
&= \int_\Omega \sum_{i=1}^n \partial_i \langle L_i x, y\rangle \,\mathrm{d}\lambda \\
&= \int_{\partial\Omega} \sum_{i=1}^n \nu_i \langle L_i x, y\rangle \,\mathrm{d}\mu \\
&= \langle L_\nu \gamma_0 x, \gamma_0 y\rangle_{\mathsf{L}^2(\partial\Omega)}.
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So for the actual differential operator of the system we have

$$\left\langle \begin{bmatrix} 0 & L_\partial \\ L_\partial^\top & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle_{\mathsf{L}^2(\Omega)} + \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} 0 & L_\partial \\ L_\partial^\top & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle_{\mathsf{L}^2(\Omega)}$$

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$$= \langle L_\nu \gamma_0 x_2, \pi_L \gamma_0 y_1 \rangle_{\mathsf{L}^2(\partial\Omega)} + \langle \pi_L \gamma_0 x_1, L_\nu \gamma_0 y_2 \rangle_{\mathsf{L}^2(\partial\Omega)}.$$

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So for the actual differential operator of the system we have

$$\left\langle \begin{bmatrix} 0 & L_\partial \\ L_\partial^\top & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle_{\mathsf{L}^2(\Omega)} + \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} 0 & L_\partial \\ L_\partial^\top & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle_{\mathsf{L}^2(\Omega)}$$

$$= \langle L_\partial x_2, y_1\rangle_{\mathsf{L}^2(\Omega)} + \langle x_2, L_\partial^\top y_1\rangle_{\mathsf{L}^2(\Omega)} + \langle L_\partial^\top x_1, y_2\rangle_{\mathsf{L}^2(\Omega)} + \langle x_1, L_\partial y_2\rangle_{\mathsf{L}^2(\Omega)}$$

$$= \langle L_\nu \gamma_0 x_2, \pi_L \gamma_0 y_1\rangle_{\mathsf{L}^2(\partial\Omega)} + \langle \pi_L \gamma_0 x_1, L_\nu \gamma_0 y_2\rangle_{\mathsf{L}^2(\partial\Omega)}.$$

We define the boundary operators $\Gamma_1 \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] := \pi_L \gamma_0 x_1$ and $\Gamma_2 \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] := L_\nu \gamma_0 x_2$.

Boundary operators

Let $A = \begin{bmatrix} 0 & L_\partial \\ L_\partial^\top & 0 \end{bmatrix}$. Then we have for $x, y \in \mathsf{H}^1(\Omega)$

$$\langle Ax, y \rangle + \langle x, Ay \rangle = \langle \Gamma_1 x, \Gamma_2 y \rangle + \langle \Gamma_2 x, \Gamma_1 y \rangle,$$

where

$$\Gamma_1 \colon x \mapsto \pi_L x_1\big|_{\partial\Omega} \quad \text{and} \quad \Gamma_2 \colon x \mapsto L_\nu \gamma_0 x_2 = \sum_{i=1}^{n} \nu_i L_i x_2\big|_{\partial\Omega}.$$

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Problem

Unfortunately, in general it is not possible to continuously extend these mappings on $\operatorname{dom} A = \mathsf{H}(L_\partial^\top, \Omega) \times \mathsf{H}(L_\partial, \Omega)$ with codomain $\mathsf{L}^2(\partial\Omega)$.

Quasi Gelfand triple

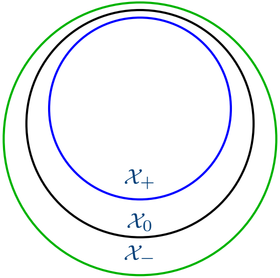


Figure: ordinary Gelfand triple

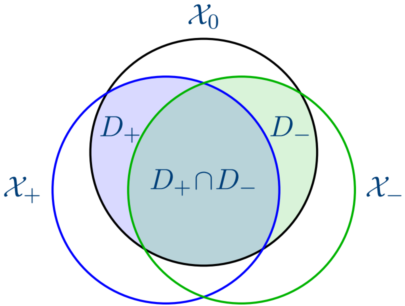
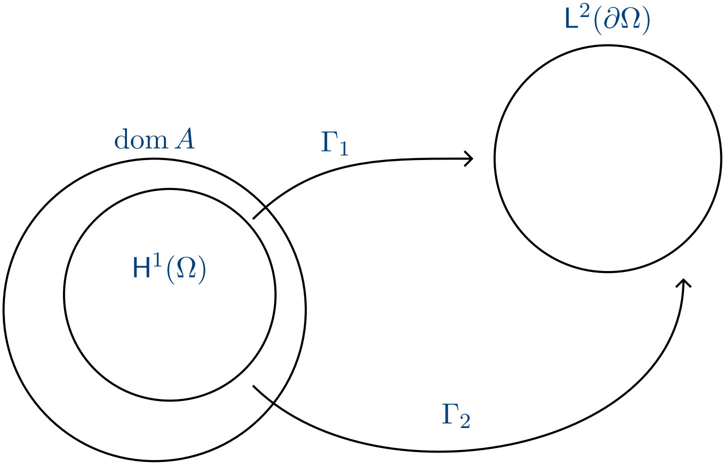


Figure: quasi Gelfand triple

$$\langle x, y \rangle_{\mathcal{X}_-,\mathcal{X}_+} = \lim_{\substack{x_n \to x \\ y_n \to y}} \langle x_n, y_n \rangle_{\mathcal{X}_0}$$

Solution via Quasi Gelfand triple



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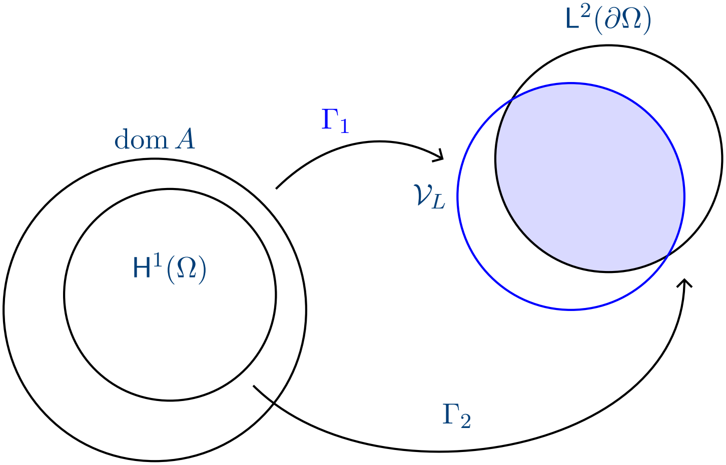

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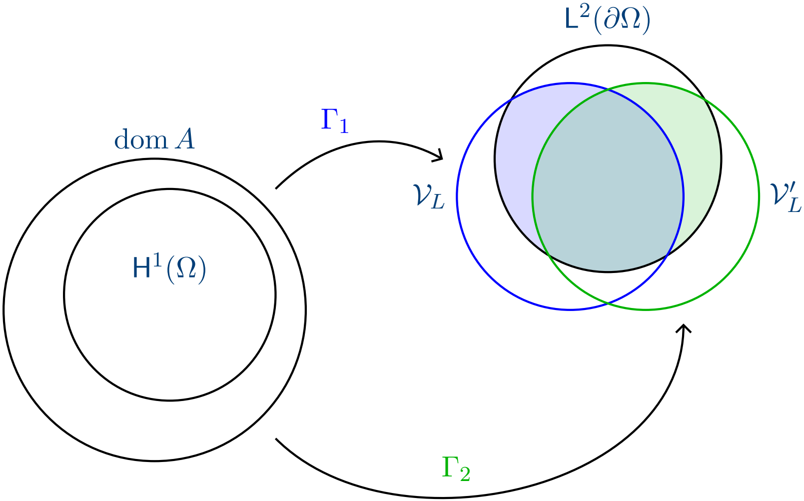

Boundary triple

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However, we can extend this abstract green identity for $x, y \in \operatorname{dom} A$

$$\langle Ax, y \rangle_{\mathsf{L}^2(\Omega)} + \langle x, Ay \rangle_{\mathsf{L}^2(\Omega)} = \langle \Gamma_1 x, \Gamma_2 y \rangle_{\mathcal{V}_L, \mathcal{V}_L'} + \langle \Gamma_2 x, \Gamma_1 y \rangle_{\mathcal{V}_L', \mathcal{V}_L}.$$

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Clearly, there exists a unitary duality mapping Ψ between $\mathcal{V}_L'$ and $\mathcal{V}_L$. Hence,

$$\langle Ax, y\rangle_{\mathsf{L}^2(\Omega)} + \langle x, Ay\rangle_{\mathsf{L}^2(\Omega)} = \langle \Gamma_1 x, \Psi\Gamma_2 y\rangle_{\mathcal{V}_L} + \langle \Psi\Gamma_2 x, \Gamma_1 y\rangle_{\mathcal{V}_L}.$$

Theorem

Let $(\mathcal{B}_+, \Gamma_1, \Psi\Gamma_2)$ *be a boundary triple for* A *such that* $(\mathcal{B}_+, \mathcal{B}_0, \mathcal{B}_-)$ *is a quasi Gelfand triple, where* Ψ *is the duality mapping between* $\mathcal{B}_+$ *and* $\mathcal{B}_-$*. Furthermore, let* $V_1, V_2 \in \mathcal{L}(\mathcal{B}_0, \mathcal{K})$*, where* $\mathcal{K}$ *is another Hilbert space. We define*

$$D := \left\{ a \in \operatorname{dom} A \,\middle|\, \Gamma_1 a, \Gamma_2 a \in \mathcal{B}_0 \text{ and } \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} a = 0 \right\}.$$

Then $A\big|_D$ *is a generator of a contraction semigroup, if*

- $\begin{bmatrix} V_1 & V_2 \end{bmatrix}\big|_{\mathcal{B}_+ \times \mathcal{B}_-}$ *is closed,*
- $\ker \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ *is dissipative i.e.* $\langle x, y \rangle \leq 0$*, if* $V_1 x + V_2 y = 0$ *and*
- $V_1 V_2^* + V_2 V_1^* \geq 0$.

Back to the pivot space

Theorem

Let $(\mathcal{V}_L, \Gamma_1, \Psi\Gamma_2)$ *be a boundary triple for* A *such that* $(\mathcal{V}_L, \mathsf{L}^2(\partial\Omega), \mathcal{V}_L')$ *is a quasi Gelfand triple, where* Ψ *is the duality mapping between* $\mathcal{V}_L$ *and* $\mathcal{V}_L'$*. Furthermore, let* $V_1, V_2 \in \mathcal{L}(\mathsf{L}^2(\partial\Omega), \mathcal{K})$*, where* $\mathcal{K}$ *is another Hilbert space. We define*

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Then $A\big|_D$ *is a generator of a contraction semigroup, if*

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Proportional Feedback

$$u(t) = L_\nu x_2(t),$$
$$\dot{x}(t) = \begin{bmatrix} 0 & L_\partial \\ L_\partial^\top & 0 \end{bmatrix} x(t),$$
$$y(t) = \pi_L x_1(t).$$

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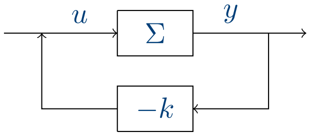

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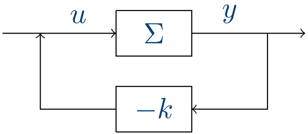

Feedback $u(t) = -ky(t)$.
This gives the domain with boundary conditions

$$\operatorname{dom} A = \left\{ x \in \mathsf{H}(L_\partial^\top, \Omega) \times \mathsf{H}(L_\partial, \Omega) \,\middle|\, L_\nu x_2 + k\pi_L x_1 = 0 \right\}$$

Example: wave equation

Let $t \in \mathbb{R}_+$ and $\Omega \subseteq \mathbb{R}^n$.

$$\begin{aligned} \frac{\partial^2}{\partial t^2} w(t,\zeta) &= \frac{1}{\rho(\zeta)} \operatorname{div} \big(T(\zeta) \nabla w(t,\zeta)\big), && \zeta \in \Omega, \\ \frac{\partial}{\partial t} w(t,\zeta) &= 0, && \zeta \in \Gamma_0. \end{aligned}$$

Choosing $x(t) = \begin{bmatrix} \rho \frac{\partial}{\partial t} w(t,\cdot) \\ \nabla w(t,\cdot) \end{bmatrix}$ leads to

$$\dot{x}(t) = \begin{bmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{bmatrix} x(t).$$

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$$\dot{x}(t) = \begin{bmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{bmatrix} x(t), \quad \text{where} \quad x(t) = \begin{bmatrix} \rho(\zeta) w(t,\cdot) \\ \nabla w(t,\cdot) \end{bmatrix}.$$

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$$\mathsf{L}^2(\Omega) \times \nabla\, \mathsf{H}^1(\Omega)$$

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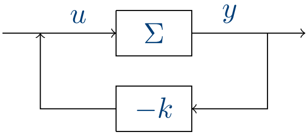
is more adequate. The domain of the differential operator is then

$$\mathsf{H}^1_{\Gamma_0}(\Omega) \times \left(\nabla \mathsf{H}^1_{\Gamma_0}(\Omega) \cap \mathsf{H}(\operatorname{div}, \Omega) \right).$$

Proportional Feedback

$$\begin{aligned} u(t) &= L_\nu x_2(t), \\ \dot{x}(t) &= \begin{bmatrix} 0 & L_\partial \\ L_\partial^\top & 0 \end{bmatrix} x(t), \\ y(t) &= \pi_L x_1(t). \end{aligned}$$



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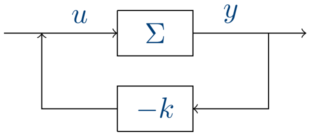

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Resolvent set

Assume $\lambda \neq 0$.

$$\left(\begin{bmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{bmatrix} - \lambda\right) x = f$$

where $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ and $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

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Hence,

$$\begin{aligned} \operatorname{div}(f_2 - \nabla x_1) + \lambda^2 x_1 &= \lambda f_1 \\ \lambda \gamma_0 x_1 + k\nu \cdot \gamma_0 \lambda x_2 &= 0 \end{aligned}$$

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Importance of state space

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{bmatrix}}^{=A} x$$

$$\operatorname{dom} A = \{x \in \mathsf{H}^1_{\Gamma_0}(\Omega) \times \mathsf{H}(\operatorname{div}, \Omega) \,|\, \nu \cdot \gamma_0 x_2 + k\gamma_0 x_1 = 0 \text{ on } \Gamma_1\}$$

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Let $\phi \in \mathsf{C}^\infty(\Omega) \setminus \{0\}$ and

$$x_1(t) \coloneqq 0, \quad x_2(t) \coloneqq \begin{bmatrix} -\partial_2 \phi \\ \partial_1 \phi \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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Then $x(t)$ is an unstable solution (constant), because

$$\operatorname{div} x_2(t) = -\partial_1\partial_2\phi + \partial_2\partial_1\phi = 0$$

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Stability

With the correct state space we get the following result

Theorem

The semigroup generated by

$$A = \begin{bmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{bmatrix}$$

$$\operatorname{dom} A = \{x \in \mathsf{H}^1_{\Gamma_0}(\Omega) \times \nabla \, \mathsf{H}^1_{\Gamma_0}(\Omega) \cap \mathsf{H}(\operatorname{div}, \Omega) \mid \nu \cdot \gamma_0 x_2 + k \gamma_0 x_1 = 0 \text{ on } \Gamma_1\}$$

is semi-uniformly stable.

Compact embedding

Theorem

Let

$$X := \nabla \mathsf{H}^1_{\Gamma_0}(\Omega) \cap \left\{ f \in \mathsf{H}(\mathrm{div}, \Omega) \,\middle|\, \nu \cdot f\big|_{\Gamma_1} \in \mathsf{L}^2(\Gamma_1) \right\},$$

$$\|f\|_X^2 := \|f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{div}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|f\|_{\mathsf{L}^2(\Gamma_1)}^2.$$

Then $X \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

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$$\|f\|_X^2 := \|f\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} f\|^2_{\mathsf{L}^2(\Omega)} + \|f\|^2_{\mathsf{L}^2(\Gamma_1)}.$$

Then $X \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in X (W.l.o.g. bounded by 1). For every $n \in \mathbb{N}$ there exists a $\phi_n \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ such that

$$\nabla \phi_n = f_n$$

By Poincaré's inequality we have

$$\|\phi_n\|_{\mathsf{L}^2} \leq C\|\nabla \phi_n\|_{\mathsf{L}^2} = C\|f_n\|_{\mathsf{L}^2} \leq C.$$

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Hence, $(\phi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathsf{H}^1(\Omega)$ and in $\mathsf{H}^{1/2}(\partial\Omega)$.
By the compact embeddings

$$\mathsf{H}^1(\Omega) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega) \quad \text{and} \quad \mathsf{H}^{1/2}(\partial\Omega) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\partial\Omega)$$

$(\phi_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathsf{L}^2(\Omega)$ and $\mathsf{L}^2(\partial\Omega)$.

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$$\begin{aligned}
&\|f_n - f_m\|^2_{\mathsf{L}^2(\Omega)} \\
&\qquad = \langle f_n - f_m, \nabla(\phi_n - \phi_m)\rangle_{\mathsf{L}^2(\Omega)} \\
&\qquad = -\langle \operatorname{div}(f_n - f_m), \phi_n - \phi_m\rangle_{\mathsf{L}^2(\Omega)} + \langle \nu \cdot (f_n - f_m), \phi_n - \phi_m\rangle_{\mathsf{L}^2(\Gamma_1)} \\
&\qquad \leq \|\operatorname{div}(f_n - f_m)\|_{\mathsf{L}^2(\Omega)}\|\phi_n - \phi_m\|_{\mathsf{L}^2(\Omega)} + \|\nu \cdot (f_n - f_m)\|_{\mathsf{L}^2(\Gamma_1)}\|\phi_n - \phi_m\|_{\mathsf{L}^2(\Gamma_1)} \\
&\qquad \leq 2\|\phi_n - \phi_m\|_{\mathsf{L}^2(\Omega)} + 2\|\phi_n - \phi_m\|_{\mathsf{L}^2(\Gamma_1)}
\end{aligned}$$

Hence, $(f_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\mathsf{L}^2(\Omega)$. □

Compact embedding

Theorem

Let

$$X := \nabla \mathsf{H}^1_{\Gamma_0}(\Omega) \cap \left\{ f \in \mathsf{H}(\mathrm{div},\Omega) \,\middle|\, \nu \cdot f\big|_{\Gamma_1} \in \mathsf{L}^2(\Gamma_1) \right\},$$

$$\|f\|_X^2 := \|f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{div}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|f\|_{\mathsf{L}^2(\Gamma_1)}^2.$$

Then $X \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

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Then $X \overset{\text{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

Note that $\mathrm{rot}\, f = \nabla \times f$. Hence

$$\mathrm{rot}\, \nabla \mathsf{H}^1_{\Gamma_0}(\Omega) = \nabla \times \nabla \mathsf{H}^1_{\Gamma_0}(\Omega) = \{0\}.$$

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Note that $\operatorname{rot} f = \nabla \times f$. Hence

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In other words $\nabla \mathsf{H}^1_{\Gamma_0}(\Omega) \subseteq \ker \operatorname{rot} \subseteq \mathsf{H}(\operatorname{rot}, \Omega)$.

Theorem

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In other words $\nabla \mathsf{H}^1_{\Gamma_0}(\Omega) \subseteq \ker \operatorname{rot} \subseteq \mathsf{H}(\operatorname{rot}, \Omega)$.
We want to generalize the theorem such that we can replace $\nabla \mathsf{H}^1_{\Gamma_0}(\Omega)$ by $\mathsf{H}(\operatorname{rot}, \Omega)$.

Compact embedding

Theorem

Let

$$X := \mathsf{H}_0(\mathrm{rot}, \Omega) \cap \mathsf{H}(\mathrm{div}, \Omega)$$
$$\|f\|_X^2 := \|f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{rot}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{div}\, f\|_{\mathsf{L}^2(\Omega)}^2.$$

Then $X \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

Compact embedding

Theorem

Let

$$X := \mathsf{H}_{\Gamma_0}(\mathrm{rot}, \Omega) \cap \mathsf{H}_{\Gamma_1}(\mathrm{div}, \Omega)$$
$$\|f\|_X^2 := \|f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{rot}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{div}\, f\|_{\mathsf{L}^2(\Omega)}^2.$$

Then $X \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

Compact embedding

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$$X := \mathsf{H}_0(\mathrm{rot}, \Omega) \cap \mathsf{H}(\mathrm{div}, \Omega)$$
$$\|f\|_X^2 := \|f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{rot}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{div}\, f\|_{\mathsf{L}^2(\Omega)}^2.$$

Then $X \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

Compact embedding

Theorem

Let

$$X := \{f \in \mathsf{H}(\mathrm{rot}, \Omega) \mid \nu \times f = 0\} \cap \mathsf{H}(\mathrm{div}, \Omega)$$

$$\|f\|_X^2 := \|f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{rot}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{div}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|\nu \times f\|_{\mathsf{L}^2(\partial\Omega)}^2.$$

Then $X \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

Compact embedding

Theorem

Let

$$X := \{ f \in \mathsf{H}(\mathrm{rot}, \Omega) \,|\, \nu \times f \in \mathsf{L}^2(\partial\Omega) \} \cap \mathsf{H}(\mathrm{div}, \Omega)$$

$$\|f\|_X^2 := \|f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{rot}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{div}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|\nu \times f\|_{\mathsf{L}^2(\partial\Omega)}^2.$$

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Then $X \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

Compact embedding

Theorem

Let

$$X := \{ f \in \mathsf{H}(\mathrm{rot}, \Omega) \,|\, \nu \times f \in \mathsf{L}^2(\Gamma_0) \} \cap \{ f \in \mathsf{H}(\mathrm{div}, \Omega) \,|\, \nu \cdot f \in \mathsf{L}^2(\Gamma_1) \}$$

$$\|f\|_X^2 := \|f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{rot}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|\mathrm{div}\, f\|_{\mathsf{L}^2(\Omega)}^2 + \|\nu \times f\|_{\mathsf{L}^2(\Gamma_0)}^2 + \|\nu \cdot f\|_{\mathsf{L}^2(\Gamma_1)}^2.$$

Then $X \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$.

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Future directions

- Higher order (e.g. Kirchhoff plate is second order)
- Stability/stabilization for general port-Hamiltonian systems
- Spatial dependency of L_i
- Differential algebraic port-Hamiltonian systems

Thank you for your attention!