Approximation for control of Port-Hamiltonian systems

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What's to come

1. Structure preserving reduced order modeling
   - Model approximation problem
   - Model approximation of Port-Hamiltonian systems with Symplectic projection
2. Rational approximation of irrational positive-real function
   - Löwner approximation
   - Spectral zeros estimation
   - Rational approximation of irrational positive-real function

Model Approximation Problem

Full-Order Model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t);\ x(0) = x_0; \\ y(t) &= Cx(t) + Du(t); \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable.

$$G(s) = C(sI_n - A)^{-1}B + D \tag{2}$$

Reduced-Order Model

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u(t);\ x_r(0) = x_{r0}; \\ y_r(t) &= C_r x_r(t) + D_r u(t); \end{aligned} \tag{3}$$

$$G_r(s) = C_r(sI_r - A_r)^{-1}B_r + D_r \tag{4}$$

G_r of order $r << n$

Projection-based approach

Petrov-Galerkin projective approximation

Two $r-$dimensional subspaces $\mathcal{V}_r, \mathcal{W}_r \subset \mathbb{R}^n$ associated with two basis matrices $V, W \in \mathbb{R}^{n \times k}$ such that $\mathcal{V}_r = \text{ran}(V)$ and $\mathcal{W}_r = \text{ran}(W)$, respectively.

$$x(t) \approx V x_r(t)$$

Residual is then constrained to be orthogonal to W.
The $r-$dimensional reduced model (3) with

$$\begin{aligned} A_r &= W^T A V,\ B_r = W^T B; \\ C_r &= CV,\ D_r = D. \end{aligned} \tag{5}$$

How to find V and W?

Proper Orthogonal Decomposition

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B \text{ with } D = 0 \tag{6}$$

Considering $U(s) = 1$

$$\begin{aligned} X(s) &= (sI - A)^{-1}B \\ Y(s) &= CX(s). \end{aligned} \tag{7}$$

State snapshot matrix

$$S = \left[X(s_1),\ X(s_2),\ \dots,\ X(x_N)\right] \in \mathbb{R}^{n\times N}. \tag{8}$$

Apply SVD as follows

$$S = Q\Sigma P^T \approx Q_r \Sigma_r P_r^T \tag{9}$$

Approximate the state as:

$$X(s) \approx Q_r \hat{X}(s), \text{ with } \hat{X}(s) \in \mathbb{R}^r \tag{10}$$

and, $A_r = Q_r^T A Q_r,\ B_r = Q_r^T B,\ C_r = CQ_r$.

Model approximation of Port-Hamiltonian System

Port-Hamiltonian System(PHS):

$$\begin{aligned} \dot{x}(t) &= (\mathbb{J}_{2n} - \mathcal{R})\mathcal{H}x(t) + Bu(t) \\ y(t) &= B^{\mathsf{T}}\mathcal{H}x(t) \end{aligned} \tag{11}$$

$x \in \mathbb{R}^{2n}$, $\mathcal{R} = \mathcal{R}^T \geq 0 \in \mathbb{R}^{2n\times 2n}$, $\mathcal{H} = \mathcal{H}^T > 0 \in \mathbb{R}^{2n\times 2n}$,

$$\mathbb{J}_{2n} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}. \tag{12}$$

Model approximation of PHS

$$G(s) = B^{\mathsf{T}}\mathcal{H}(sI_{2n} - (\mathbb{J}_{2n} - \mathcal{R})\mathcal{H})^{-1}B. \tag{13}$$

$$G_r(s) = \hat{B}^{\mathsf{T}}\hat{\mathcal{H}}(sI_{2k} - (\mathbb{J}_{2k} - \hat{\mathcal{R}})\hat{\mathcal{H}})^{-1}\hat{B}. \tag{14}$$

$$\begin{aligned} \dot{z}(t) &= (\mathbb{J}_{2k} - \hat{\mathcal{R}})\hat{\mathcal{H}}z(t) + \hat{B}u(t) \\ \hat{y}(t) &= \hat{B}^{\mathsf{T}}\hat{\mathcal{H}}z(t), \end{aligned} \tag{15}$$

Problem: POD do not preserve the port-Hamiltonian structure

Symplectic Model Approximation

Symplectic Vector Space

Let, $\mathbb{V}$ be a vector space of dimension $2n$. A symplectic form on $\mathbb{V}$ is a bilinear mapping $\Omega : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ that is

1. Antisymmetric: $\Omega(v, \bar{v}) = -\Omega(\bar{v}, v)$, for all $v, \bar{v} \in \mathbb{V}$
2. Non-degenerate: $\Omega(v, \bar{v}) = 0$, for all $v \in \mathbb{V}$ if and only if $\bar{v} = 0$

The pair $(\mathbb{V}, \Omega)$ is called a symplectic vector space.

Example

The canonical example of a symplectic vector space is $\mathbb{R}^{2n}$ with the bilinear form $\Omega(v_1, v_2) = \langle v_1, \mathbb{J}_{2n} v_2 \rangle \; \forall v_1, v_2 \in \mathbb{R}^{2n}$ where $\langle ., . \rangle$ is the Euclidean inner product and $\mathbb{J}_{2n}$ is the Possion matrix.

Symplectic Projection

Symplectic Map

Let $(\mathbb{R}^{2n}, \Omega)$ and $(\mathbb{R}^{2m}, \Omega)$ be two symplectic vector spaces with $m < n$. Let $Q : \mathbb{R}^{2m}$ be a linear mapping. It is called a symplectic map and Q a symplectic matrix w. r. t. Ω if

$$Q^T \mathbb{J}_{2n} Q = \mathbb{J}_{2m}. \tag{16}$$

The set of all $2n \times 2m$ symplectic matrices is called the symplectic Stiefel manifold, denoted by $Sp(2m, \mathbb{R}^{2n})$.

Symplectic Inverse

For each symplectic matrix $Q \in \mathbb{R}^{2n \times 2m}$, the symplectic inverse, denoted by Q^{-l}, is defined by

$$Q^{-l} = \mathbb{J}_{2m}^{\mathsf{T}} Q^{\mathsf{T}} \mathbb{J}_{2n} \ \in \mathbb{R}^{2m \times 2n}. \tag{17}$$

Symplectic Projection

Lemma (Peng & Mohseni(2016))

Suppose $Q \in Sp(2m, \mathbb{R}^{2n})$ and Q^{-l} is the symplectic inverse of Q as defined in (17). Then,

(i) $Q^{-l}Q = I_{2m}$;

(ii) $Q^{-l}\mathbb{J}_{2n} = \mathbb{J}_{2m}Q^{\mathsf{T}}$;

(iii) If $v \in \mathrm{ran}(Q)$, then $v = QQ^{-l}v$;

(iv) Q^{-l} is also Symplectic;

Symplectic Approximation of PHS

$$\begin{bmatrix} (sI_{2n} - \mathbb{J}_{2n}\mathcal{H}) & -B \\ B^{\mathsf{T}}\mathcal{H} & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}, \tag{18}$$

in which u is fixed, i.e., $y = G(s)u$.

Theorem

Consider a $Q \in Sp(2m, \mathbb{R}^{2n})$ together with its symplectic inverse Q^{-l} and obtained the reduced model:

$$\begin{bmatrix} (sI_{2k} - \mathbb{J}_{2k}\hat{\mathcal{H}}) & -\hat{B} \\ \hat{B}^{\mathsf{T}}\hat{\mathcal{H}} & 0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}. \tag{19}$$

where, $\hat{B} = Q^{-l}B$ and $\hat{\mathcal{H}} = Q^{-l}\mathcal{H}Q$. If $x_0 \in \text{ran}(Q)$ and $\text{Im}(B) \subseteq \text{ran}(Q)$ then the reduced system preserve the port-Hamiltonian structure.

Symplectic Basis Generation

Let, $x_{0l} \in \mathbb{R}^{2n} (l = 1, \dots, N)$ denote N data points.

$$S_x = [x_{01}, x_{02}, \dots, x_{0N}].$$

$$S_z = Q^{-l} S_x = [z_{01}, z_{02}, \dots, z_{0N}] \in \mathbb{R}^{2k \times N} \tag{20}$$

where, $x = Qz$ with $Q \in \mathbb{R}^{2n \times 2k}$, $Q^T J_{2n} Q = J_{2k}$, and $Q^{-l} Q = I_{2k}$.

Proper Symplectic Decomposition(PSD) and POD

$$\begin{aligned} &\min \ \|S_x - QQ^{-l} S_x\|_F. \\ &\text{sub. to } \ Q^T J_{2n} Q = J_{2k} \\ &\& \ Q^{-l} = \mathbb{J}_{2k}^T Q^T \mathbb{J}_{2n} \end{aligned} \tag{21}$$

$$\begin{aligned} &\min \ \|S_x - QQ^T S_x\|_F \\ &\text{sub. to } \ Q^T Q = I_{2k} \end{aligned} \tag{22}$$

Theorem

Suppose, $S_x = \begin{bmatrix} S_{x1} \\ S_{x2} \end{bmatrix} \in \mathbb{R}^{2n\times N}$ is the snapshot matrix where $S_{x1},\ S_{x2}\ \in \mathbb{R}^{n\times N}$. Construct $S_1 \in \mathbb{R}^{n\times 2N}$ from S_x as follows:

$$S_1 = \begin{bmatrix} S_{x1} & S_{x2} \end{bmatrix}.$$

Construct a symplectic matrix $Q_1 = \text{diag}(\Phi, \Phi) \in \mathbb{R}^{2n\times 2k}$ where $\Phi \in \mathbb{R}^{n\times k}$ is the POD basis of the snapshot matrix S_1. Then the optimization problem (OP1)

$$\begin{aligned} &\min\ \|S_x - Q_1 Q_1^{-l} S_x\|_F \\ &\text{s. t. } Q_1^T \mathbb{J}_{2n} Q_1 = \mathbb{J}_{2k} \text{ and } Q^{-l} = \mathbb{J}_{2k}^T Q^T \mathbb{J}_{2n} \end{aligned} \tag{23}$$

is equivalent to the optimization problem (OP2)

$$\begin{aligned} &\min\ \|S_1 - \Phi\Phi^T S_1\|_F \\ &\text{s.t., } \Phi^T\Phi = I_k \end{aligned} \tag{24}$$

PHS with Dissipation

Lemma(Son et. al.(2020))

For $Q, T \in Sp(2k, \mathbb{R}^{2n})$, ran($Q$) = ran($T$) if and only if there exists a matrix $K \in OrSp(2n)$ such that $T = KQ$

Lemma

If $T \in Sp(2k, \mathbb{R}^{2n})$, then $(T^{-l})^T \in\in Sp(2k, \mathbb{R}^{2n})$ with $T^{-l} = \mathbb{J}_{2k}^T Q^T K^T \mathbb{J}_{2n}$

PHS with Dissipation

Theorem

Consider a $T \in Sp(2m, \mathbb{R}^{2n})$ together with its symplectic inverse T^{-l} and obtained the reduced model:

$$\begin{bmatrix} (sI_{2k} - (\mathbb{J}_{2k} - \hat{\mathcal{R}})\hat{\mathcal{H}}) & -\hat{B} \\ \hat{B}^{\mathsf{T}}\hat{\mathcal{H}} & 0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}. \tag{25}$$

where, $\hat{B} = T^{-l}B$, $\hat{\mathcal{R}} = T^{-l}\mathcal{R}(T^{-l})^T$, and $\hat{\mathcal{H}} = T^{-l}\mathcal{H}(T^{-l})^T$. If $x_0 \in$ ran(Q), Im$(\mathcal{H}) \subseteq$ ran(Q) and Im$(B) \subseteq$ ran(Q) then the reduced system preserve the port-Hamiltonian structure.

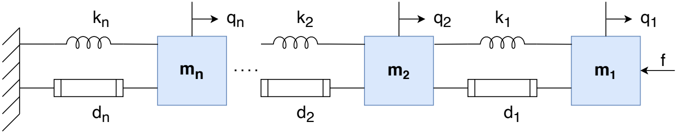

Figure: n-coupled Mass-Spring-Damper system

Numerical Example

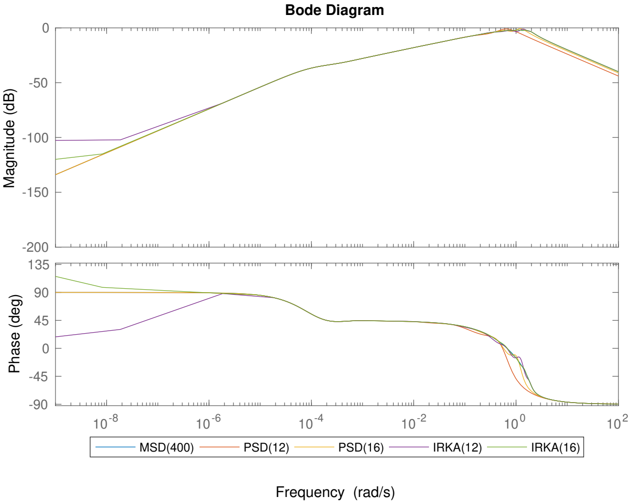

Figure: Nyquist plotof original and reduced order system

Numerical Example

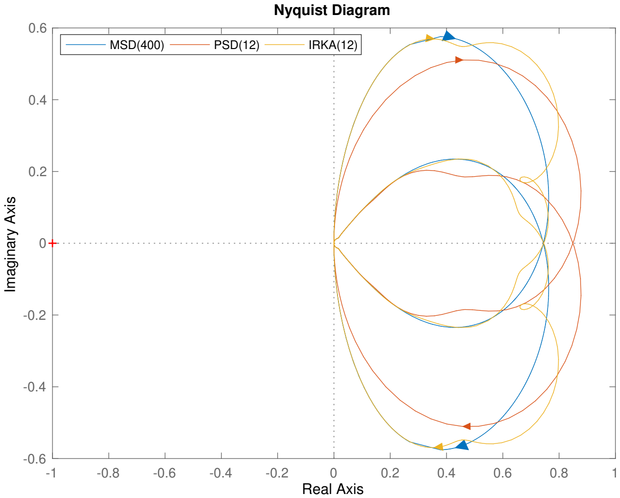

Figure: Nyquist plotof original and reduced order system

Rational approximation of irrational positive real function

Given: $(x_i, f_i), \ i = 1, \ldots, n$ where $x_i \in \mathbb{C}$ and $f_i = H(x_i) \in \mathbb{C}$

Objective: Find a positive real function $\hat{H}$ such that

$$\hat{H}(s) = C(sE - A)^{-1}B + D$$

with state-space dimension $m(\leq n)$ that well reproduces the data *i.e.,* $\hat{H}(x_i) = f_i, \ \forall i$.

system is passive $\Longleftrightarrow$ its transfer function is positive real

Problem Formulation: For the provided data set (x_i, f_i), is it possible to construct a positive real function $\hat{H}(x)$ satisfying the interpolation conditions $\hat{H}(x_i) = f_i$,?

Löwner Framework: Algorithm

1. Split data set:

$$[x_1, x_2, \ldots, x_N] = [\mu_1, \mu_2, \ldots, \mu_{\underline{n}}] \cup [\lambda_1, \lambda_2, \ldots, \lambda_{\overline{n}}]$$
$$[f_1, f_2, \ldots, f_N] = [\nu_1, \nu_2, \ldots, \nu_{\underline{n}}] \cup [\omega_1, \omega_2, \ldots, \omega_{\overline{n}}]$$

2. Compute the Löwner and the shifted Löwner matrices:

$$\mathbb{L}_{ij} = \frac{(\nu_i - \omega_j)}{(\mu_i - \lambda_j)}, \quad [\mathbb{L}_s]_{ij} = \frac{(\mu_i \nu_i - \lambda_j \omega_j)}{\mu_i - \lambda_j}$$
$$V = [\nu_1, \ldots\ldots, \nu_{\underline{n}}]^T, \quad W = [\omega_1, \ldots\ldots, \omega_{\overline{n}}]$$

3. Realization: $E = -\mathbb{L},\ A = -\mathbb{L}_s, B = V, C = W$
   Trade-off between accuracy of fit and complexity of the model

$$\begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \epsilon \end{bmatrix} \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} = s\mathbb{L} - \mathbb{L}_s$$

$$E = -Y_1^* \mathbb{L} X_1,\ A = -Y_1^* \mathbb{L}_s X_1,\ C = W X_1,\ B = Y_1^* V$$

Löwner Framework: Passivity

Nevanlinna - Pick interpolation Problem

$$\Pi = \begin{bmatrix} \frac{f_1+f_1^*}{x_1+x_1^*} & \cdots & \frac{f_1+f_n^*}{x_1+x_n^*} \\ \vdots & \cdots & \vdots \\ \frac{f_n+f_1^*}{x_n+x_1^*} & \cdots & \frac{f_n+f_n^*}{x_n+x_n^*} \end{bmatrix}$$

must be non-negative definite to construct a positive real rational function

Mirror image of the given data i.e., $(-x_i^*, -f_i^*), i = 1, 2, \ldots, N$.

$$\Pi = \begin{bmatrix} \frac{f_1-f_1^*}{x_1-x_1^*} & \cdots & \frac{f_1-f_n^*}{x_1-x_n^*} \\ \vdots & \cdots & \vdots \\ \frac{f_n-f_1^*}{x_n-x_1^*} & \cdots & \frac{f_n-f_n^*}{x_n-x_n^*} \end{bmatrix} \Longrightarrow \Pi = \mathbb{L}$$

Spectral Zeros

The spectral zeros of a positive real transfer function $H(s)$ with realization (E, A, B, C, D) are the complex numbers $s_z \in \mathbb{C}$ such that

$$G(s_z) = H(s_z) + H^\mathsf{T}(-s_z) = 0. \tag{26}$$

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & E^\mathsf{T} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & -A^\mathsf{T} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ -C^\mathsf{T} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} C & B^\mathsf{T} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + (D + D^\mathsf{T}) \end{aligned} \tag{27}$$

Lemma

1. *Spectral zeros have mirror images in the complex plane with respect to imaginary axis.*
2. *For strictly passive system, no spectral zeros are on the imaginary axis.*

Spectral Zeros

Lemma

1. Spectral zeros have mirror images in the complex plane with respect to imaginary axis.
2. For strictly passive system, no spectral zeros are on the imaginary axis.

Theorem

If a set of spectral zeros in the closed right-half complex plane of the original strictly passive system are selected as interpolation points, then the constructed Löwner matrix (Pick matrix) is positive definite.

Problem: Spectral zeros of the system is unknown

Spectral Zeros Estimation

Theorem

A state transformation between two minimal realizations of a linear system does not change the spectral zeros of the system.

We use Löwner approximation twice:

1. 1st Löwner approximation : Provide a minimal realization, most likely not positive real, used to compute spectral zeros
2. 2nd Löwner approximation : Provide a positive real realization

Motivating Example

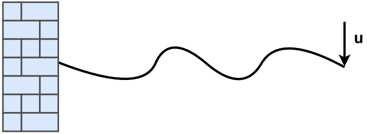

Figure: Vibrating string

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial \xi^2} + \zeta \frac{\partial^3 \omega}{\partial t \partial \xi^2}; \quad (28)$$

$$\begin{aligned} \omega(0,t) &= 0; \\ \frac{\partial \omega}{\partial \xi}(1,t) &= u(t); \\ \frac{\partial \omega}{\partial t}(1,t) &= y(t); \end{aligned}$$

Transfer function:

$$H(s) = \frac{1}{\sqrt{c^2 + s\zeta}} \frac{\sinh(\sqrt{r})}{\cosh(\sqrt{r})}, \text{ where } r = \frac{s^2}{c^2 + s\zeta} \quad (29)$$

Irrational $\Longrightarrow$ Approximate by an rational one

Löwner Framework: Approximation

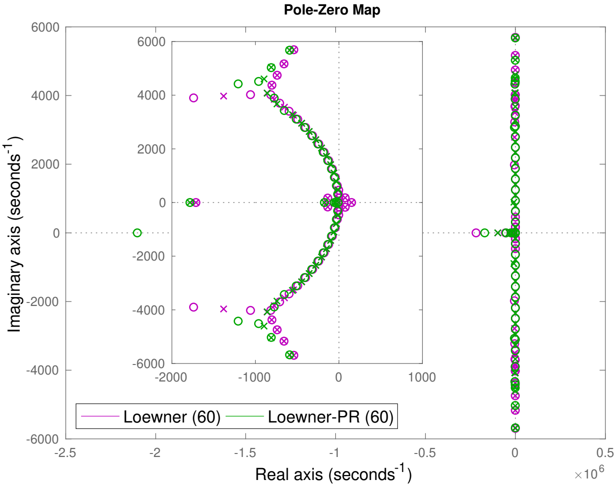

Figure: Poles and Zeros of stable and unstable Loewner realization

Löwner Framework: Approximation

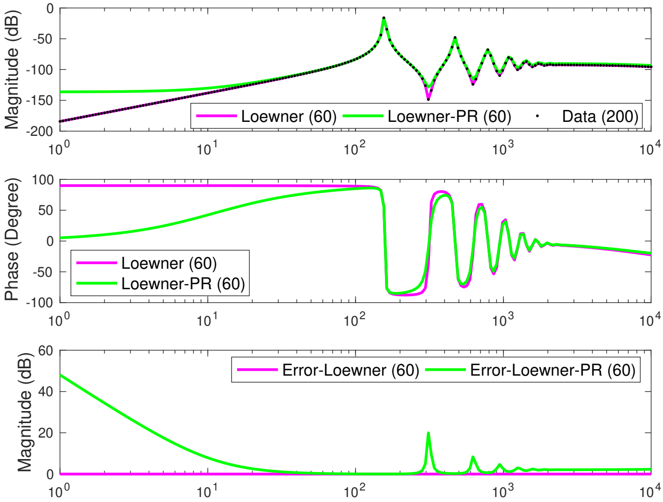

Figure: Bode Plot

Löwner Framework: Approximation

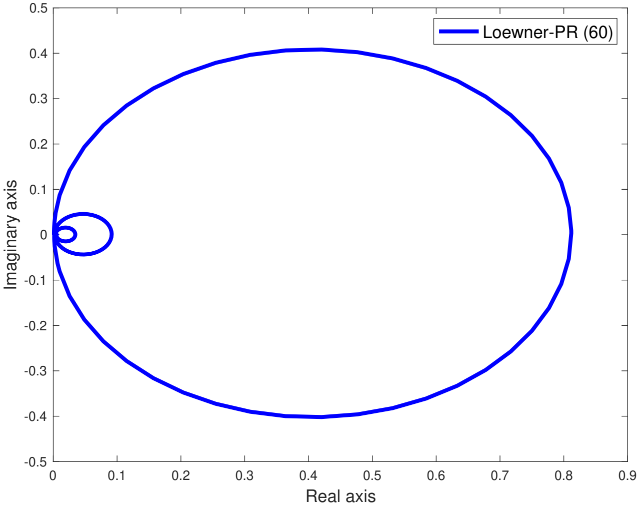

Figure: Nyquist Plot

Publications

1. Structure Preserving MOR of Port-Hamiltonian system in Frequency domain (Preparing)
2. Data-driven rational approximation of irrational positive-real function (Preparing)

Thank You!