Geometrically exact beams (in networks)

well-posedness, stabilization, control

Charlotte Rodriguez – ESR 5

Final ConFlex meeting – IMB Bordeaux

August 4, 2021

ConFlex
www.conflex.org

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 765579.

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

Information

- Supervisor Pr. Günter Leugering
- Fellowship with ConFlex: July 1, 2018 – June 30, 2021 (ended).

- Now: short term contract at FAU Erlangen-Nürnberg.
- Deliverable D 2.4 due for September 30, 2021.
- Future plans: academia (candidate for a MSC two year fellowship at Imperial) or industry (research scientist).

Publications/preprints

[A1] Charlotte Rodriguez, and Günter Leugering. "Boundary feedback stabilization for the intrinsic geometrically exact beam model".

In: *SIAM Journal on Control and Optimization* 58 (6), pp. 3533–3558 (2020).

`DOI`: `10.1137/20M1340010`.

[A2] Charlotte Rodriguez. "Networks of geometrically exact beams: well-posedness and stabilization".

In: *Mathematical Control and Related Fields* (2021).

`DOI`: `10.3934/mcrf.2021002`. Advance online publication.

[A3] Günter Leugering, Charlotte Rodriguez, and Yue Wang. "Nodal profile control for networks of geometrically exact beams".

In: *Journal de Mathématiques Pures et Appliquées* (2021).

`DOI`: `10.1016/j.matpur.2021.07.007`. In press.

[P1] Marc Artola, Charlotte Rodriguez, Andrew Wynn, Rafael Palacios and Günter Leugering. "Optimisation of Region of Attraction Estimates for the Exponential Stabilisation of the Intrinsic Geometrically Exact Beam Model".

In: *IEEE Conference on Decision and Control* (2021).

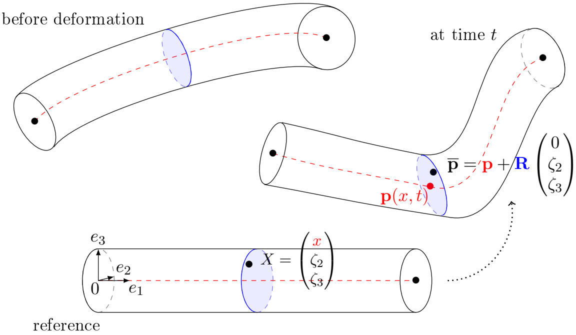
Accepted.

[P2] Günter Leugering, Charlotte Rodriguez, and Yue Wang. "Exact controllability of networks of elastic strings springs and masses".

In preparation (2021).

Presentation of the model

"Geometrically exact beam"
"Nonlinear Timoshenko beam"
"Geometrically nonlinear beam"



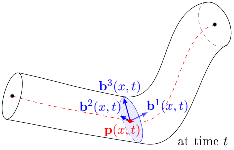
Small strains BUT large motions.

linear constitutive law $\leftarrow$ $\rightarrow$ nonlinear governing system

Presentation of the model

Framework 1. The state is $(\mathbf{p}, \mathbf{R})$, expressed in some fixed coordinate system $\{e_j\}_{j=1}^3$,

- centerline's position $\mathbf{p}(x,t) \in \mathbb{R}^3$
- cross sections' orientation given by the columns $\mathbf{b}^j$ of $\mathbf{R}(x,t) \in \mathrm{SO}(3)$



Set in $(0,\ell) \times (0,T)$, the governing system reads (freely vibrating beam)

$$\boxed{\begin{bmatrix} \partial_t & \mathbf{0} \\ (\partial_t \widehat{\mathbf{p}}) & \partial_t \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{M} v \right] = \begin{bmatrix} \partial_x & \mathbf{0} \\ (\partial_x \widehat{\mathbf{p}}) & \partial_x \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} z \right]}$$

given $\mathbf{M}(x), \mathbf{C}(x) \in \mathbb{S}^6_{++}$ the mass and flexibility matrices and $\Upsilon_c(x) \in \mathbb{R}^3$ the curvature before deformation, and where v, s depend on $(\mathbf{p}, \mathbf{R})$:

$$v = \begin{bmatrix} \mathbf{R}^\intercal \partial_t \mathbf{p} \\ \mathrm{vec}\left(\mathbf{R}^\intercal \partial_t \mathbf{R}\right) \end{bmatrix}, \qquad z \quad = \mathbf{C}^{-1} \begin{bmatrix} \mathbf{R}^\intercal \partial_x \mathbf{p} - e_1 \\ \mathrm{vec}\left(\mathbf{R}^\intercal \partial_x \mathbf{R}\right) - \Upsilon_c \end{bmatrix}.$$

Notation. Cross-product: $\widehat{u}\,\zeta = u \times \zeta$ and $\mathrm{vec}\,(\widehat{u}) = u$
$\mathrm{SO}(3)$: rotation matrices. $\mathbb{S}^n_{++}$: positive definite symmetric matrices of size n.

Presentation of the model

Framework 2. The state is $y = \begin{bmatrix} v \\ z \end{bmatrix}$, expressed in the moving basis $\{\mathbf{b}^j\}_{j=1}^3$,

- linear and angular velocities $v(x,t) \in \mathbb{R}^6$
- internal forces and moments $z(x,t) \in \mathbb{R}^6$

Set in $(0,\ell) \times (0,T)$, the governing system reads (freely vibrating beam)

$$\boxed{\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \partial_t y - \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \partial_x y - \begin{bmatrix} \multicolumn{2}{c}{\mathbf{0}} & \widehat{\Upsilon}_c & \mathbf{0} \\ & & \widehat{e}_1 & \widehat{\Upsilon}_c \\ \widehat{\Upsilon}_c & \widehat{e}_1 & \multicolumn{2}{c}{\mathbf{0}} \\ \mathbf{0} & \widehat{\Upsilon}_c & & \end{bmatrix} y = - \begin{bmatrix} \widehat{v}_2 & \mathbf{0} & \mathbf{0} & \widehat{z}_1 \\ \widehat{v}_1 & \widehat{v}_2 & \widehat{z}_1 & \widehat{z}_2 \\ \multicolumn{2}{c}{\mathbf{0}} & \widehat{v}_2 & \widehat{v}_1 \\ & & \mathbf{0} & \widehat{v}_2 \end{bmatrix} \begin{bmatrix} \mathbf{M}v \\ \mathbf{C}z \end{bmatrix}}$$

denoting by v_1, z_1 and v_2, z_2 the first and last 3 components of v, z.

We will also write the governing system:

$$\boxed{\partial_t y + A(x)\partial_x y + \overline{B}(x) y = \overline{g}(x,y).}$$

Notation. Cross-product: $\widehat{u}\,\zeta = u \times \zeta$ and $\mathrm{vec}\,(\widehat{u}) = u$

Presentation of the model

Two frameworks:

1. GEB. Quasilinear
second-order
(Reissner '81, Simo '85)
'Wave-like'

linked by a nonlinear transformation:

$$\mathcal{T} \colon (\mathbf{p}, \mathbf{R}) \longmapsto \begin{bmatrix} \mathbf{I}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{R}^\intercal \partial_t \mathbf{p} \\ \mathrm{vec}\left(\mathbf{R}^\intercal \partial_t \mathbf{R}\right) \\ \mathbf{R}^\intercal \partial_x \mathbf{p} - e_1 \\ \mathrm{vec}\left(\mathbf{R}^\intercal \partial_x \mathbf{R}\right) - \Upsilon_c \end{bmatrix} = \begin{bmatrix} v \\ z \end{bmatrix}$$

2. IGEB. Semilinear (quadratic)
first-order hyperbolic
(Hodges '03)
'Hamiltonian framework' (Simo '88)

Networks:

The states are now $(\mathbf{p}_i, \mathbf{R}_i)_{i\in\mathcal{I}}$ and $(y_i)_{i\in\mathcal{I}}$. Recall that $y_i = \begin{bmatrix} v_i \\ z_i \end{bmatrix}$.

Transmission conditions at a multiple node n (where several beams meet):

- Rigid joint. Any two incident beams i, j remain attached to each other $\mathbf{p}_i = \mathbf{p}_j$ and without changing the respective angles between them $\mathbf{R}_i R_i^\intercal = \mathbf{R}_j R_j^\intercal$.
- Kirchhoff condition. In the fixed basis, the internal forces and moments exerted by the incident beams at the node are balanced with the external load.

$\rightarrow$ derive the corresponding transmission conditions for the IGEB model:

- Continuity of velocities. For any two incident beams i, j,

$$\boxed{\begin{bmatrix} R_i & \mathbf{0} \\ \mathbf{0} & R_i \end{bmatrix} v_i = \begin{bmatrix} R_j & \mathbf{0} \\ \mathbf{0} & R_j \end{bmatrix} v_j}$$

- Corresponding Kirchhoff condition. For q_n the external load applied at the node n, expressed in the body-attached basis,

$$\boxed{\sum_{\text{incident beam } i} \tau_i^n \begin{bmatrix} R_i & \mathbf{0} \\ \mathbf{0} & R_i \end{bmatrix} z_i = q_n}$$

Well-posedness for networks

Boundary condition at a simple node n: for the incident beam i,

$$\boxed{\tau_i^n z_i = q_n\,, \quad \text{or} \quad v_i = q_n}$$

Based on results on results on abstract one-dimensional first-order hyperbolic systems, Li-Jin '01 and Bastin-Coron '16 and '17:

- $(y_i)_{i\in\mathcal{I}} \in \prod_{i=1}^N C^1([0,\ell_i]\times[0,T];\mathbb{R}^{12})$ semi-global in time
- $(y_i)_{i\in\mathcal{I}} \in C^0([0,T), \prod_{i\in\mathcal{I}} H^k(0,\ell_i;\mathbb{R}^{12}))$ local in time, with $q_n = -K_n v_i$ where $K_n \in \mathbb{R}^{6\times 6}$

Require some regularity from the data/coefficients and some properties of the transmission conditions for the system in diagonal form.

Assumption 1

Let $m \in \{1,2,\dots\}$ be given. For all $i \in \mathcal{I}$, we suppose that

- $\mathbf{C}_i, \mathbf{M}_i \in C^m([0,\ell_i];\mathbb{S}^6_{++})$;
- for $\Theta_i := (\mathbf{C}_i^{1/2}\mathbf{M}_i\mathbf{C}_i^{1/2})^{-1}$, there exists $U_i, D_i \in C^m([0,\ell_i];\mathbb{R}^{6\times 6})$ s.t. $\Theta_i = U_i^\intercal D_i^2 U_i$ in $[0,\ell_i]$, where $D_i(x) \in \mathbb{S}^6_{++}$ diagonal & consists of the square roots of the eigenvalues of $\Theta_i(x)$, and $U_i(x)$ is unitary.

Inverting the transformation

Two frameworks:

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second-order
(Reissner '81, Simo '85)
'Wave-like'

linked by a nonlinear transformation:

$$\mathcal{T} \colon (\mathbf{p}, \mathbf{R}) \longmapsto \begin{bmatrix} \mathbf{I}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{R}^\intercal \partial_t \mathbf{p} \\ \mathrm{vec}\left(\mathbf{R}^\intercal \partial_t \mathbf{R}\right) \\ \mathbf{R}^\intercal \partial_x \mathbf{p} - e_1 \\ \mathrm{vec}\left(\mathbf{R}^\intercal \partial_x \mathbf{R}\right) - \Upsilon_c \end{bmatrix} = \begin{bmatrix} v \\ z \end{bmatrix}$$

2. IGEB. Semilinear (quadratic)
first-order hyperbolic
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'Hamiltonian framework' (Simo '88)

Inverting the transformation

The GEB model

$$\begin{cases} \begin{bmatrix} \partial_t & \mathbf{0} \\ (\partial_t \widehat{\mathbf{p}}) & \partial_t \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{M} v \right] = \begin{bmatrix} \partial_x & \mathbf{0} \\ (\partial_x \widehat{\mathbf{p}}) & \partial_x \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} z \right] & \text{in } (0,\ell)\times(0,T) \quad (1a) \\ (\mathbf{p}, \mathbf{R})(0,t) = (f^{\mathbf{p}}, f^{\mathbf{R}}) & t \in (0,T) \quad (1b) \\ z(\ell,t) = -Kv(\ell,t) & t \in (0,T) \quad (1c) \\ (\mathbf{p}, \mathbf{R})(x,0) = (\mathbf{p}^0, \mathbf{R}^0)(x) & x \in (0,\ell) \quad (1d) \\ (\partial_t \mathbf{p}, \mathbf{R}W)(x,0) = (\mathbf{p}^1, w^0)(x) & x \in (0,\ell), \quad (1e) \end{cases}$$

and its IGEB counterpart

$$\begin{cases} \partial_t y + A(x)\partial_x y + \overline{B}(x) y = \overline{g}(x,y) & \text{in } (0,\ell)\times(0,T) \quad (2a) \\ v(0,t) = \mathbf{0} & \text{for } t \in (0,T) \quad (2b) \\ z(\ell,t) = -Kv(\ell,t) & \text{for } t \in (0,T) \quad (2c) \\ y(x,0) = y^0(x) & \text{for } x \in (0,\ell). \quad (2d) \end{cases}$$

Inverting the transformation

The GEB model

$$\begin{cases} \begin{bmatrix} \partial_t & \mathbf{0} \\ (\partial_t \widehat{\mathbf{p}}) & \partial_t \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{M} v \right] = \begin{bmatrix} \partial_x & \mathbf{0} \\ (\partial_x \widehat{\mathbf{p}}) & \partial_x \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} z \right] & \text{in } (0,\ell)\times(0,T) \quad \text{(1a)} \\ (\mathbf{p},\mathbf{R})(0,t) = (f^{\mathbf{p}}, f^{\mathbf{R}}) & t \in (0,T) \quad \text{(1b)} \\ z(\ell,t) = -Kv(\ell,t) & t\in(0,T) \quad \text{(1c)} \\ (\mathbf{p},\mathbf{R})(x,0) = (\mathbf{p}^0, \mathbf{R}^0)(x) & x\in(0,\ell) \quad \text{(1d)} \\ (\partial_t \mathbf{p}, \mathbf{R}W)(x,0) = (\mathbf{p}^1, w^0)(x) & x \in (0,\ell), \quad \text{(1e)} \end{cases}$$

and its IGEB counterpart

$$\begin{cases} \partial_t y + A(x)\partial_x y + \overline{B}(x) y = \overline{g}(x,y) & \text{in } (0,\ell)\times(0,T) \quad \text{(2a)} \\ v(0,t) = \mathbf{0} & \text{for } t\in(0,T) \quad \text{(2b)} \\ z(\ell,t) = -Kv(\ell,t) & \text{for } t \in (0,T) \quad \text{(2c)} \\ y(x,0) = y^0(x) & \text{for } x\in(0,\ell). \quad \text{(2d)} \end{cases}$$

The transformation $\mathcal{T}\colon E_1 \to E_2$ is well defined for

$$E_1 = \left\{ (\mathbf{p},\mathbf{R}) \in C^2\Big([0,\ell]\times[0,T];\mathbb{R}^3 \times \mathrm{SO}(3)\Big) : \text{(1b), (1d) hold} \right\}$$

$$E_2 = \left\{ y \in C^1\Big([0,\ell]\times[0,T];\mathbb{R}^{12}\Big) : \text{(3) holds for } u := \mathrm{diag}(\mathbf{I}_6, \mathbf{C})y \right\},$$

where

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{p}^0 = \mathbf{R}^0(u_3(\cdot,0)+e_1), \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{R}^0 = \mathbf{R}^0(\widehat{u}_4(\cdot,0) + \widehat{\Upsilon}_c), \quad \text{in } (0,\ell) \tag{3a}$$

$$\partial_t \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} - \partial_x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} \widehat{\Upsilon}_c & \widehat{e}_1 \\ \mathbf{0} & \widehat{\Upsilon}_c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \widehat{u}_2 & \widehat{u}_1 \\ \mathbf{0} & \widehat{u}_2 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}, \quad \text{in } (0,\ell)\times(0,T) \tag{3b}$$

$$u_1(0,\cdot) = \mathbf{0}, \quad u_2(0,\cdot) = \mathbf{0}, \quad \text{in } (0,T), \tag{3c}$$

where we use the notation $u = (u_1^\intercal, \ldots, u_4^\intercal)^\intercal$ with $u_k(x,t) \in \mathbb{R}^3$ for all $k \in \{1,2,3,4\}$.

Inverting the transformation

Main step:

Lemma

If $(f^{\mathbf{p}}, f^{\mathbf{R}}) \in \mathbb{R}^3 \times \mathrm{SO}(3)$ *and* $(\mathbf{p}^0, \mathbf{R}^0) \in C^2([0,\ell]; \mathbb{R}^3 \times \mathrm{SO}(3))$ *satisfy*

$$(f^{\mathbf{p}}, f^{\mathbf{R}}) = (\mathbf{p}^0, \mathbf{R}^0)(0), \tag{4}$$

then the transformation $\mathcal{T}\colon E_1 \to E_2$ *is bijective.*

Inverting the transformation

Theorem

Assume that (4) *holds with*

$$\mathbf{M}, \mathbf{C} \in C^1([0,\ell];\mathbb{R}^{6\times 6})$$
$$R \in C^2([0,\ell];\mathrm{SO}(3))$$
$$(\mathbf{p}^0, \mathbf{R}^0) \in C^2([0,\ell];\mathbb{R}^3\times\mathrm{SO}(3))$$
$$\mathbf{p}^1, w^0 \in C^1([0,\ell];\mathbb{R}^3)$$

$$y^0 := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{-1} \end{bmatrix} \begin{bmatrix} (\mathbf{R}^0)^\intercal \mathbf{p}^1 \\ (\mathbf{R}^0)^\intercal w^0 \\ (\mathbf{R}^0)^\intercal \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{p}^0 - e_1 \\ \mathrm{vec}\big((\mathbf{R}^0)^\intercal \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{R}^0\big) - \Upsilon_c \end{bmatrix}.$$

Then,
if there exists a unique solution $y \in C^1([0,\ell]\times[0,T];\mathbb{R}^{12})$ *to* (2) *with initial data* y^0 *(for some* $T>0$*),*

$\Longrightarrow$ *there exists a unique solution* $(\mathbf{p},\mathbf{R}) \in C^2([0,\ell]\times[0,T];\mathbb{R}^3\times\mathrm{SO}(3))$ *to* (1) *with initial data* $(\mathbf{p}^0,\mathbf{R}^0,\mathbf{p}^1,w^0)$ *and boundary data* $(f^{\mathbf{p}}, f^{\mathbf{R}})$*, and* $y = \mathcal{T}(\mathbf{p},\mathbf{R})$*.*

Idea of the proof.

- A solution y to (2) always belongs to E_2 due to the last six governing equations in (2a), the last six initial conditions in (2d) and the Dirichlet conditions (2b) and since we maintained the link between the initial and boundary data of (1) and (2).

  $\Rightarrow$ Previous Lemma automatically provides $(\mathbf{p},\mathbf{R})$, candidate to be solution (1);

- $(\mathbf{p},\mathbf{R})$ then fulfills the governing system of the GEB model;
- The rest of boundary and initial conditions of System (2) lead to those of (1);
- Uniqueness comes from that of the IGEB model and the fact that $\mathcal{T}$ is bijective.

Inverting the transformation

Two frameworks:

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2. IGEB. Semilinear (quadratic)
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Inverting the transformation

Idea of the proof of the Lemma:

Given $y \in E_2$, $\exists!$ $(\mathbf{p}, \mathbf{R}) \in E_1$ such that $\mathcal{T}(\mathbf{p}, \mathbf{R}) = y$?

Step 1. $\mathcal{T}(\mathbf{p}, \mathbf{R}) = y \Leftrightarrow$ PDE system.

$$\boxed{\begin{cases} \partial_t \mathbf{R} = \mathbf{R}\widehat{u}_2 & \text{in } (0,\ell)\times(0,T) \\ \partial_x \mathbf{R} = \mathbf{R}(\widehat{u}_4 + \widehat{\Upsilon}_c) & \text{in } (0,\ell)\times(0,T) \\ \mathbf{R}(0,0) = \mathbf{R}^0(0) \end{cases}}, \qquad \begin{cases} \partial_t \mathbf{p} = \mathbf{R}u_1 & \text{in } (0,\ell)\times(0,T) \\ \partial_x \mathbf{p} = \mathbf{R}(u_3 + e_1) & \text{in } (0,\ell)\times(0,T) \\ \mathbf{p}(0,0) = \mathbf{p}^0(0). \end{cases}$$

Step 2. Quaternions. $\mathbf{R} = (q_0^2 - \langle q\,, q\rangle)\mathbf{I}_3 + 2qq^\intercal + 2q_0\widehat{q} \quad \leftrightarrow \quad \mathbf{q} = \begin{bmatrix} q_0 \\ q \end{bmatrix}, \ \|\mathbf{q}\| \equiv 1$

Lemma

Let $f \in C^1([0,\ell]\times[0,T];\mathbb{R}^3)$ and let z represent either of the spatial or time variables x, t. The function $\mathbf{q} \in C^1([0,\ell]\times[0,T];\mathbb{R}^4)$ fulfills both $|\mathbf{q}| \equiv 1$ and

$$\partial_z \mathbf{q} = \mathcal{U}(f)\mathbf{q}, \quad \textit{in } (0,\ell)\times(0,T), \qquad \textit{with} \quad \mathcal{U}(f) := \tfrac{1}{2}\begin{bmatrix} 0 & -f^\intercal \\ f & -\widehat{f} \end{bmatrix}$$

if and only if the map $\mathbf{R} \in C^1([0,\ell]\times[0,T];\mathrm{SO}(3))$ parametrized by $\mathbf{q}$ fulfills

$$\partial_z \mathbf{R} = \mathbf{R}\widehat{f}, \quad \textit{in } (0,\ell)\times(0,T).$$

Thus, the first system is equivalent to ($\mathbf{R}^0(0)$ parametrized by $\mathbf{q}_{\mathrm{in}}$)

$$\boxed{\begin{cases} \partial_t \mathbf{q} = \mathcal{U}(u_2)\mathbf{q} & \text{in } (0,\ell)\times(0,T) \\ \partial_x \mathbf{q} = \mathcal{U}(u_4 + \Upsilon_c)\mathbf{q} & \text{in } (0,\ell)\times(0,T) \\ \mathbf{q}(0,0) = \mathbf{q}_{\mathrm{in}}. \end{cases}}$$

Step 3. Seemingly overdetermined systems.
The last three equations in (3b) are equivalent to:

$$\mathcal{U}(u_2)\mathcal{U}(u_4+\Upsilon_c)-\mathcal{U}(u_4+\Upsilon_c)\mathcal{U}(u_2)+\partial_x(\mathcal{U}(u_2))-\partial_t(\mathcal{U}(u_4+\Upsilon_c))=0,$$

hence they provide compatibility conditions to solve for $\mathbf{q}$ by means of the following lemma.

Lemma

Let $A, B \in C^1([0,\ell]\times[0,T];\mathbb{R}^{n\times n})$ *be such that* $AB-BA+(\partial_x A)-(\partial_t B)=0$ *holds in* $(0,\ell)\times(0,T)$*. Then,*

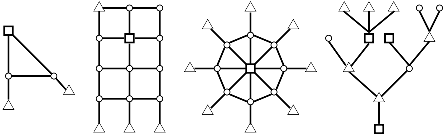
$$\begin{cases} \partial_t y = Ay & \text{in } (0,\ell)\times(0,T) \\ \partial_x y = By & \text{in } (0,\ell)\times(0,T) \\ y(0,0)=y_{\mathrm{in}}. \end{cases}$$

admits a unique solution $y\in C^1([0,\ell]\times[0,T];\mathbb{R}^n)$*, for any given* $y_{\mathrm{in}}\in\mathbb{R}^n$*.*

Then, we inject the obtained $\mathbf{R}$ in the second system, and solve for $\mathbf{p}$ using (3c) together with the first three equations in (3b).

Control of nodal profiles:

Square are the "charged nodes", where the state should meet some profiles
Triangles are the "controlled nodes".



Travelling time:

Let us denote the eigenvalues of A_i by $\{\lambda_i^k\}_{k=1}^{12}$ (there are negative and positive eigenvalues).

We may define, for any $i \in \mathcal{I}$, the function $\Lambda_i \in C^0([0, \ell_i]; (0, +\infty))$ and the travelling time $T_i > 0$ by

$$\Lambda_i(x) = \left| \min_{k \in \{1,\dots,12\}} \frac{1}{\lambda_i^k(x)} \right| \quad \text{and} \quad T_i = \int_0^{\ell_i} \Lambda_i(x) dx;$$

Exact controllability of nodal profiles

Theorem

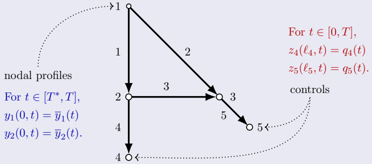
Consider the A-shaped network. Suppose that $R_i \in C^2([0,\ell_i];\mathrm{SO}(3))$ *and Assumption 1* $(m=2)$ *holds. Then, for any*

$$T > T^* > \max\{T_1, T_2\} + \max\{T_4, T_5\} =: \overline{T}.$$

there exists $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_0)$, *for some* $\delta, \gamma > 0$, *and*

(i) for all initial - boundary data $y_i^0 \in C^1([0,\ell_i];\mathbb{R}^{12})$ *and* $q_n \in C^1([0,T];\mathbb{R}^6)$ *satisfying the first-order compatibility conditions and* $\|y_i^0\|_{C_x^1} + \|q_n\|_{C_t^1} \leq \delta$, *and*

(ii) for all nodal profiles $\overline{y}_1, \overline{y}_2 \in C^1([T^*,T];\mathbb{R}^{12})$, *satisfying* $\|\overline{y}_i\|_{C_t^1} \leq \gamma$ *and the transmission conditions at the node* $n=1$,



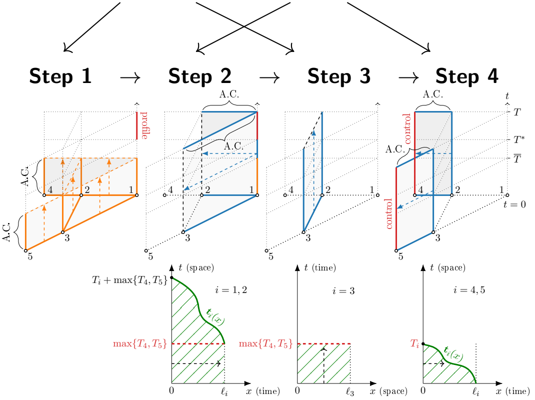
there exist controls $q_4, q_5 \in C^1([0,T];\mathbb{R}^6)$ *with* $\|q_i\|_{C_t^1} \leq \varepsilon$, *such that the IGEB network admits a unique solution* $(y_i)_{i\in\mathcal{I}} \in \prod_{i=1}^N C^1([0,\ell_i]\times[0,T];\mathbb{R}^{12})$, *which fulfills* $\|y_i\|_{C_x^1} \leq \varepsilon$ *and*

$$y_i(0,t) = \overline{y}_i(t) \quad \textit{for all } i \in \{1,2\},\ t \in [T^*,T].$$

Constructive method of by Li and collaborators; notably here Zhuang '18 and '21.

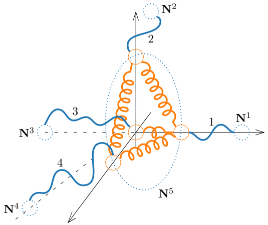
Exact controllability of nodal profiles

$$\partial_t y_i + A_i \partial_x y_i + \overline{B}_i y_i = \overline{g}(x, y_i) \qquad \partial_x y_i + A_i^{-1}\partial_t y_i + A_i^{-1}\overline{B}_i y_i = A_i^{-1}\overline{g}_i(\cdot, y_i)$$

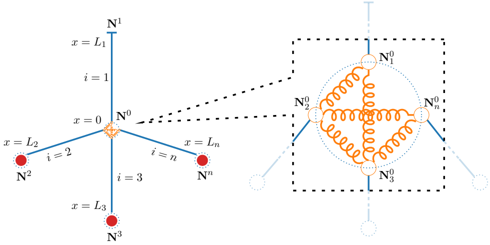


$$\mathbf{t}_i(x) = T_i + \max\{T_4, T_5\} + \int_0^x \underbrace{\min_{1\le k\le 12} \frac{1}{\lambda_i^k(\xi)}}_{=-\Lambda_i(\xi)} d\xi, \qquad \mathbf{t}_i(x) = T_i - \int_0^x \Lambda_i(\xi) d\xi,$$

String-spring-mass network



Nonlinear strings coupled to an elastic body in $\mathbb{R}^3$. The resulting network has four simple nodes $\{\mathbf{N}_i\}_{i=1}^4$ and one multiple nodes $\mathbf{N}_5$, itself expended into a network of springs.



Outlook

- More general junction conditions for networks of geometrically exact beams: mass-spring junction.
- Numerics: test the effect of the feedback in numerical simulations.
- Well-posedness and stabilization with Kelvin-Voigt damping. Relax the smallness assumption on the initial data.
- Stabilization of star-shaped network: removing one control.
- Nodal profile control: theorem with general conditions sufficient for obtaining nodal profile controllability for any network.

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Thank you for your attention! Questions?

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This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 765579.

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