PI Control of MIMO Stable Nonlinear Plants Using Projected Dynamical Systems Theory

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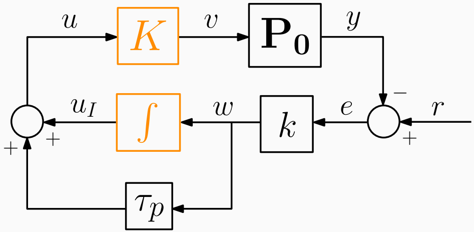
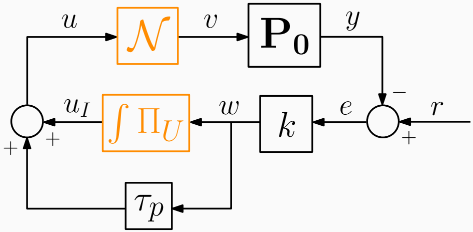
Classical PI control loop



Saturating PI control loop



where $y, r, e, w, u_I, u \in \mathbb{R}^p$, $v \in \mathbb{R}^m$, $k, \tau_p > 0$, $K \in \mathbb{R}^{m \times p}$.

Keywords Anti-Windup, Singular Perturbations, Projected Dynamical Systems.

Projected Dynamical Systems

Operator $\Pi_U(u, w)$ - Definition

Definition 1.* For a closed convex set $U \subset \mathbb{R}^p$, let the projection operator $P_U(w)$ be defined for all $w \in \mathbb{R}^p$ as

$$P_U(w) = \arg\min_{u \in U} \|w - u\|,$$

then we define $\Pi_U(u, w)$ for all $w \in \mathbb{R}^p$ and $u \in U$ as

$$\Pi_U(u, w) = \lim_{\delta \to 0} \frac{(P_U(u + \delta w) - u)}{\delta}.$$

*P. Dupuis. Large deviations analysis of reflected diffusions and constrained stochastic approximation algorithms in convex sets. *Stochastics*, 21(1):63-96, 1987.

Operator $\Pi_U(u, w)$ - Operating Principle

Lemma 1.* Let the operator $\Pi_U(u, w)$ and the set U be as defined before. Then

1. If $u \in U^{\circ}$, then $\Pi_U(u, w) = w$.
2. if $u \in \partial U$, then $\Pi_U(u, w) = w + \beta(u) n^*(u)$, where

$$n^*(u) = \arg\max_{n \in n(u)} \langle w, -n \rangle, \quad \text{and} \quad \beta(u) = \max\{0, \langle w, -n^*(u) \rangle\},$$

*P. Dupuis. Large deviations analysis of reflected diffusions and constrained stochastic approximation algorithms in convex sets. *Stochastics*, 21(1):63-96, 1987.

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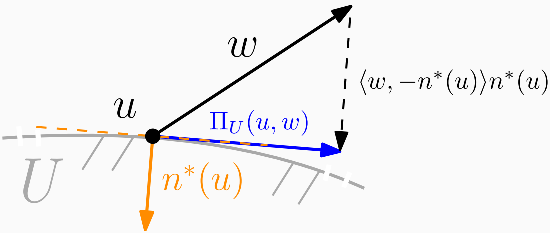

*P. Dupuis. Large deviations analysis of reflected diffusions and constrained stochastic approximation algorithms in convex sets. *Stochastics*, 21(1):63-96, 1987.

Projected Dynamical System - Definition

Definition 2.* Let $U \subset \mathbb{R}^p$ be a closed and convex set, and $F : U \to \mathbb{R}^p$ a vector field. Define a projected dynamical system PDS(F, U) as the map $\Phi : U \times \mathbb{R} \mapsto U$, such that $\phi_{u_0}(t) = \Phi(u_0, t)$ is a Carathéodory solution of

$$\dot{\phi}_{u_0}(t) = \Pi_U(\phi_{u_0}(t), -F(\phi_{u_0}(t))), \quad \phi_{u_0}(0) = u_0. \tag{1}$$

*A. Nagurney and D. Zhang. *Projected Dynamical Systems and Variational Inequalities with Applications*. International Series in Operations Research & Management Science. Springer US, 1996.

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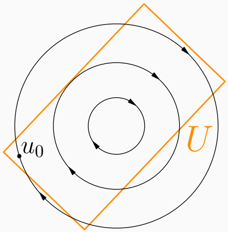
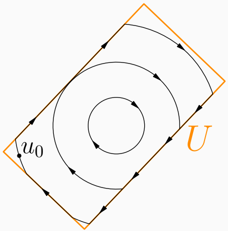




*A. Nagurney and D. Zhang. *Projected Dynamical Systems and Variational Inequalities with Applications*. International Series in Operations Research & Management Science. Springer US, 1996.

Projected Dynamical System - Existence and Uniqueness

Theorem 1.* Let $U \subset \mathbb{R}^p$ be a closed and convex set. Assume that there exists a finite $B > 0$ such that the vector field $-F : U \to \mathbb{R}^p$ satisfies:

$$\|F(u)\| \leq B(1 + \|u\|) \;\; \forall\, u \in U,$$
$$\langle -F(u_1) + F(u_2), u_1 - u_2 \rangle \leq B\|u_1 - u_2\|^2 \;\; \forall\, u_1, u_2 \in U.$$

Then:

1. For any $u_0 \in U$, there exists a unique solution $u : [0, \infty) \to U$ to the initial value problem (1).
2. If $u_n \to u_0$ as $n \to \infty$, then $u(t; u_n)$ converges to $u(t; u_0)$ uniformly on every compact set in $[0, \infty)$.

A. Nagurney and D. Zhang. *Projected Dynamical Systems and Variational Inequalities with Applications*. International Series in Operations Research & Management Science. Springer US, 1996.

Control Problem Formulation

The nonlinear plant $\mathbf{P_0}$ to be controlled is described by:

$$\dot{x} = f_0(x, v), \qquad y = g(x),$$

with $f_0 \in C^2(\mathbb{R}^n \times \mathcal{V}; \mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n; \mathbb{R}^p)$, and $\mathcal{V} \subset \mathbb{R}^m$ $(m \geq p)$ open domain.

Control Objective

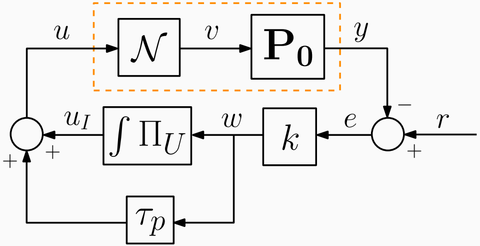
The control objective is to make the output signal y track a constant reference signal $r \in Y \subset \mathbb{R}^p$, while making sure that the input signal v converges to a steady-state value in a desired compact set $V \subset \mathbb{R}^m$ (e.g., determined by operational constraints).

Closed-Loop System - Equations

The closed-loop system is described by

$$\dot{x} = f_0(x, \mathcal{N}(u_I + \tau_p k(r - g(x)))), \quad \dot{u}_I = \Pi_U(u_I, k(r - g(x))), \tag{2}$$

where $\mathcal{U} \subset \mathbb{R}^p$ is an open domain, $U \subset \mathcal{U}$ is compact and convex, $\mathcal{N} \in C^2(\mathcal{U}, \mathcal{V})$, $V = \mathcal{N}(U)$, $k > 0$ and $\tau_p \geq 0$, with state space $\mathbb{R}^n \times \mathcal{U}$.



*P. Lorenzett and G. Weiss. "PI Control of stable nonlinear plants using projected dynamical systems theory," 2021.

Notation. Denote by

$$\mathcal{D}_r := \left\{ \left[\begin{smallmatrix} x \\ u_I \end{smallmatrix}\right] \in \mathbb{R}^n \times \mathcal{U} \mid u_I + \tau_p k(r - g(x)) \in \mathcal{U} \right\} .$$

Proposition 1. Consider the closed-loop system (2), with $k, \tau_p \in \mathbb{R}$, $r \in \mathbb{R}^p$. Then for every $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}_r$ with $u_0 \in U$, there exists $\tau \in (0, \infty]$ such that (2), with initial conditions $z(0) = \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]$, has a unique Carathéodory solution (or state trajectory) z defined on $[0, \tau)$. If τ is finite and maximal (i.e., the state trajectory cannot be continued beyond τ), then $\limsup_{t \to \tau} \|z(t)\| = \infty$, or

$$\lim_{t \to \tau} (u_I(t) + \tau_p k(r - g(x(t)))) \in \partial\mathcal{U} .$$

Closed-Loop Stabilty Analysis

Assumption 1. There exists a function $\Xi \in C^1(\mathcal{V}; \mathbb{R}^n)$ such that

$$f_0(\Xi(v), v) = 0 \qquad \forall\, v \in \mathcal{V}.$$

Moreover, the set of equilibrium points $\{\Xi(v) \,\big|\, v \in \mathcal{V}\}$ is uniformly exponentially stable. This means that there exist $\varepsilon_0 > 0$, $\lambda > 0$ and $\rho \geq 1$ such that for each constant input $v_0 \in \mathcal{V}$, the following holds:

If $\|x(0) - \Xi(v_0)\| \leq \varepsilon_0$, then for every $t \geq 0$,

$$\|x(t) - \Xi(v_0)\| \leq \rho e^{-\lambda t} \|x(0) - \Xi(v_0)\|.$$

Notation. Let $G(v) := g(\Xi(v)) \in C^1(\mathcal{V}; \mathbb{R}^p)$ denote the steady-state input-output map corresponding to $\mathbf{P_0}$.

Assumption 2. The plant $\mathbf{P_0}$ satisfies Assumption 1. Moreover, there exist an open set $\mathcal{U} \subset \mathbb{R}^p$, a function $\mathcal{N} \in C^2(\mathcal{U}, \mathcal{V})$, and $\mu > 0$ such that

$$\langle G(\mathcal{N}(u_1)) - G(\mathcal{N}(u_2)), u_1 - u_2 \rangle \geq \mu \|u_1 - u_2\|^2$$

for all $u_1, u_2 \in \mathcal{U}$, i.e., $G \circ \mathcal{N}$ is strictly monotone.

Assumption 2

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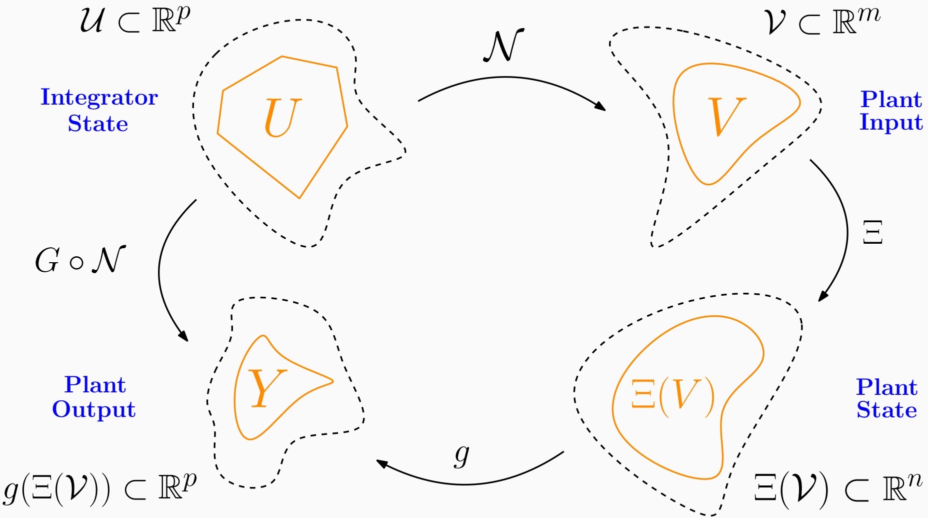
for all $u_1, u_2 \in \mathcal{U}$, i.e., $G \circ \mathcal{N}$ is strictly monotone.

We denote $Y = G(\mathcal{N}(U))$, and, for any $r \in Y$, we define

$$u_r := (G \circ \mathcal{N})^{-1}(r) \qquad x_r := \Xi(\mathcal{N}(u_r)).$$


$\mathcal{U} \subset \mathbb{R}^p$
Integrator State
U
$\mathcal{N}$
$\mathcal{V} \subset \mathbb{R}^m$
V
Plant Input
$G \circ \mathcal{N}$
Ξ
Y
Plant Output
g
$\Xi(V)$
Plant State
$g(\Xi(\mathcal{V})) \subset \mathbb{R}^p$
$\Xi(\mathcal{V}) \subset \mathbb{R}^n$

Main Stability Theorem

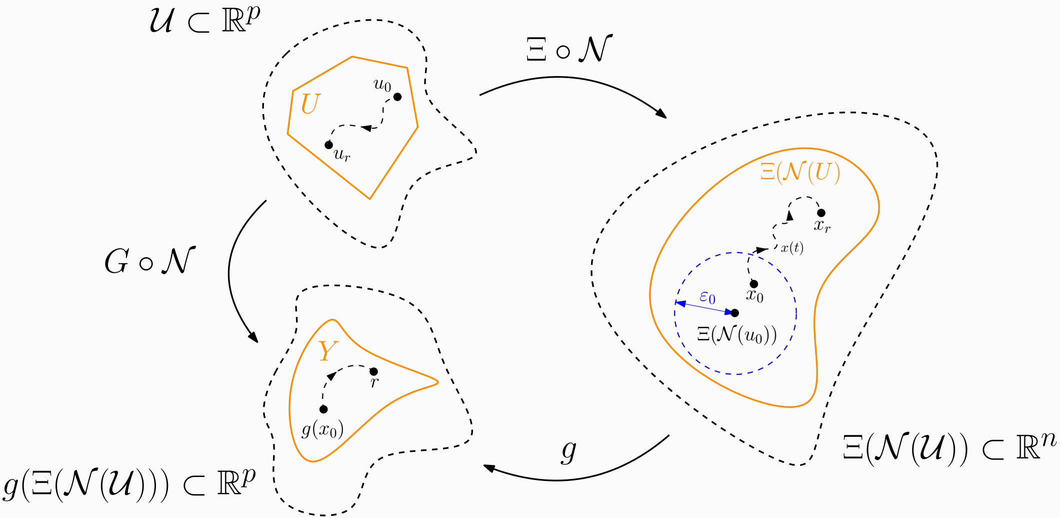
Theorem 2. Consider the closed-loop system (2), where $\mathbf{P_0}$ satisfies Assumption 2. Then there exists a $\kappa > 0$ such that if the gain $k \in (0, \kappa]$, then for any $r \in Y = G(\mathcal{N}(U))$, $(\Xi(\mathcal{N}(u_r)), u_r)$ is a (locally) exponentially stable equilibrium point of the closed-loop system (2), with state space $X = \mathbb{R}^n \times \mathcal{U}$. If the initial state $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}_r$, of the closed-loop system satisfies $u_0 \in U$ and $\|x_0 - \Xi(\mathcal{N}(u_0))\| \leq \varepsilon_0$, then

$$x(t) \to \Xi(\mathcal{N}(u_r)), \qquad u_I(t) \to u_r, \qquad y(t) \to r,$$

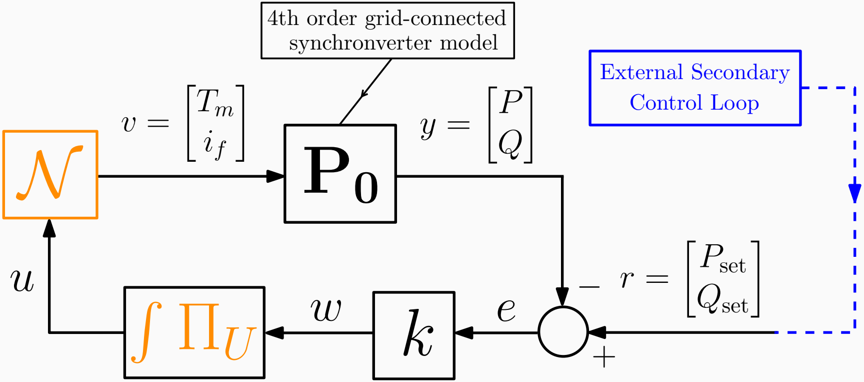
and this convergence is at an exponential rate.

*P. Lorenzett and G. Weiss. "PI Control of stable nonlinear plants using projected dynamical systems theory," 2021.

Intuition of the Result



Numerical Example



where $m = p = 2$, and $n = 4$.

Fourth Order Grid-Connected Synchronverter Model

The plant $\mathbf{P_0}$, with state $x = [\ i_d\ i_q\ \omega\ \delta\ ]^\top \in \mathbb{R}^4$, is described by the equations*

$$H\dot{x} = A(x,v)x + h(x,v), \quad y = g(x),$$

with

$$H = \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad h(x,v) = \begin{bmatrix} V\sin\delta \\ V\cos\delta \\ T_m + D_p\omega_n \\ -\omega_g \end{bmatrix},$$

$$A(x,v) = \begin{bmatrix} -R & \omega L & 0 & 0 \\ -\omega L & -R & -mi_f & 0 \\ 0 & mi_f & -D_p & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad g(x) = -V \begin{bmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} i_q \\ i_d \end{bmatrix}.$$

*P. Lorenzetti, Z. Kustanovich, S. Shivratri, and G. Weiss. "The equilibrium points and stability of grid-connected synchronverters," *IEEE Trans. Power Systems*, to appear in 2021.

Parameters' Choice

Parameter	Numerical Value
ω_g	100π rad/sec (50 Hz)
V	$230\sqrt{3}$ Volts
J	0.2 Kg·m^2/rad
D_p	3 N·m/(rad/sec)
R	$1.875\,\Omega$
L	56.75 mH
m	3.5 H
ω_n	100π rad/sec (50 Hz)

Assumption 1 - The Mapping Ξ

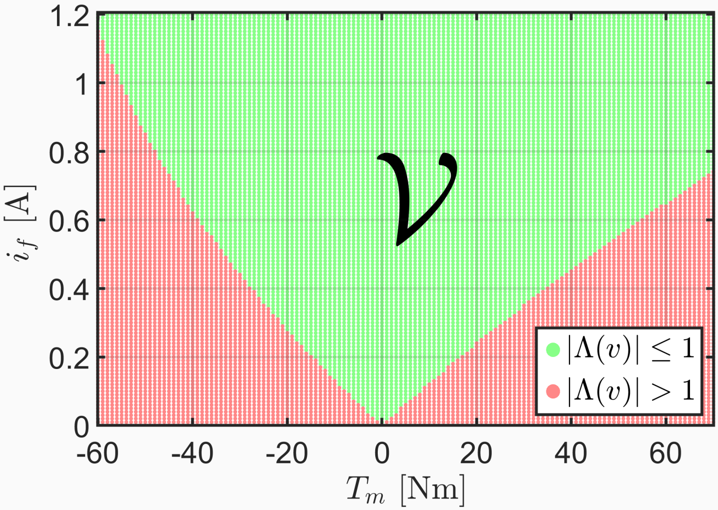
The function $\Xi : \mathcal{V} \to \mathbb{R}^4$ is given by

$$\Xi(v) = \begin{bmatrix} -\frac{T_m \omega_g}{m i_f p} + \frac{V \sin(\arccos \Lambda(v) - \phi)}{R} \\ -\frac{T_m}{m i_f} \\ \omega_g \\ \arccos \Lambda(v) - \phi \end{bmatrix}, \qquad \text{where}$$

$$\phi \in \left(0, \frac{\pi}{2}\right) \quad \text{s. t.} \quad \tan\phi = \frac{\omega_g L}{R}, \quad \Lambda(v) = -\frac{T_m}{m i_f} \frac{L\sqrt{p^2 + \omega_g^2}}{V} + \frac{m i_f \omega_g p}{V\sqrt{p^2 + \omega_g^2}},$$

$$p = \frac{R}{L}, \quad \text{and} \quad \mathcal{V} = \{(T_m, i_f) \in \mathbb{R} \times (0, \infty) \,\big|\, |\Lambda(v)| \le 1\}.$$

V. Natarajan and G. Weiss. "Almost global asymptotic stability of a grid-connected synchronous generator," *Math. of Control, Signals and Systems*, vol. 30, 2018.


1.2
1
0.8
0.6
0.4
0.2
0
-60
-40
-20
0
20
40
60
i_f [A]
T_m [Nm]
$\mathcal{V}$
$|\Lambda(v)| \leq 1$
$|\Lambda(v)| > 1$

ConFlex

We choose $\mathcal{N} = G_{\text{right}}^{-1} \in C^2(G(\mathcal{V}), \mathcal{V})$, described by the equation

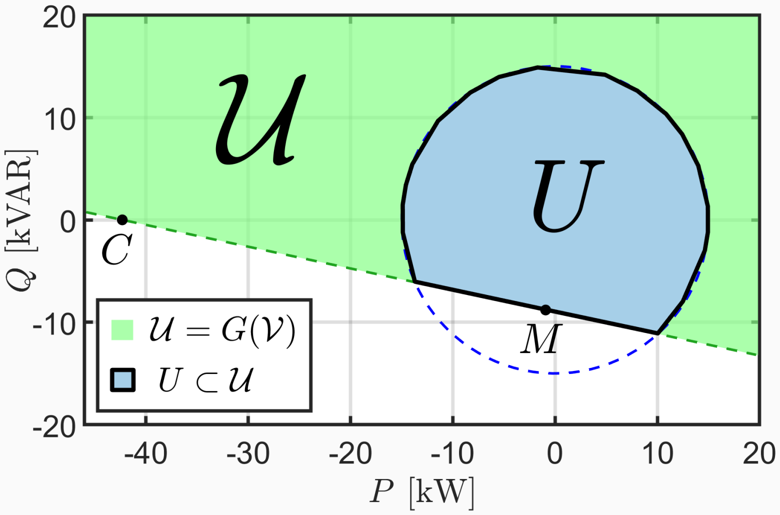
$$\mathcal{N}(u) = \begin{bmatrix} \frac{4R^2\|u-C\|^2-V^4}{4V^2\omega_g R} \\ \frac{\|u-M\|\|Z\|}{V\omega_g m} \end{bmatrix},$$

where

$$C = \begin{bmatrix} -\frac{V^2}{2R} \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} R \\ \omega_g L \end{bmatrix}, \quad M = -\frac{V^2}{\|Z\|^2} Z,$$

so that Assumption 2 is satisfied $(G \circ \mathcal{N} = I)$ with $\mathcal{U} = G(\mathcal{V})$.

Assumption 2 - The sets $\mathcal{U}$ and U



Assumption 2 - The Mapping G (Alternative Choice)

An alternative choice to $\mathcal{N} = G_{\text{right}}^{-1}$ could be, e.g., $\mathcal{N} = K \in \mathbb{R}^{2\times 2}$ given by

$$K = \begin{bmatrix} \frac{1}{50} & 0 \\ 0 & \frac{1}{5000} \end{bmatrix}. \tag{3}$$

A. Nagurney and D. Zhang. *Projected Dynamical Systems and Variational Inequalities with Applications*. International Series in Operations Research & Management Science. Springer US, 1996.

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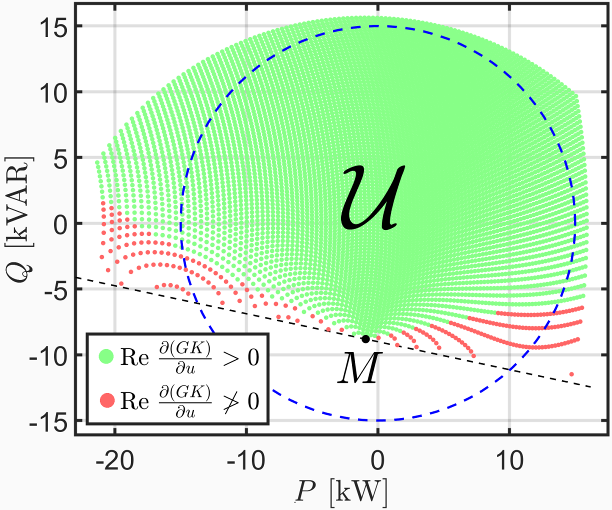
The (strict) monotonicity of $G \circ \mathcal{N} \in C^1(\mathcal{U}, \mathbb{R}^p)$ is equivalent to the fact that $\mathrm{Re}\,\frac{\partial(G\circ\mathcal{N})}{\partial u}$ is strongly positive, i.e., there exists a $\mu > 0$ such that

$$\left\langle \frac{\partial(G\circ\mathcal{N})}{\partial u} w, w \right\rangle \geq \mu \|w\|^2 \quad \forall\, w \in \mathbb{R}^p,\ \forall\, u \in \mathcal{U}.$$

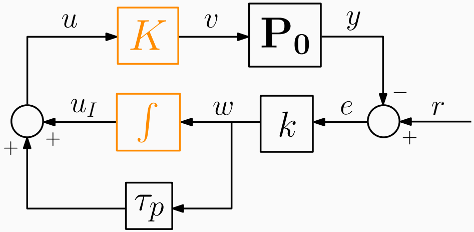
A. Nagurney and D. Zhang. *Projected Dynamical Systems and Variational Inequalities with Applications*. International Series in Operations Research & Management Science. Springer US, 1996.

Assumption 2 - The sets $\mathcal{U}$ and U (Alternative Choice)

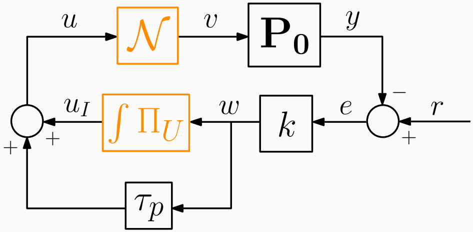


Numerical Results - The Comparison

Classical PI control loop

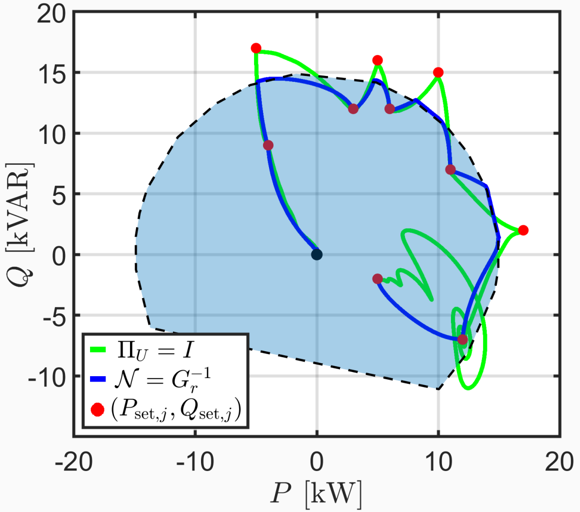


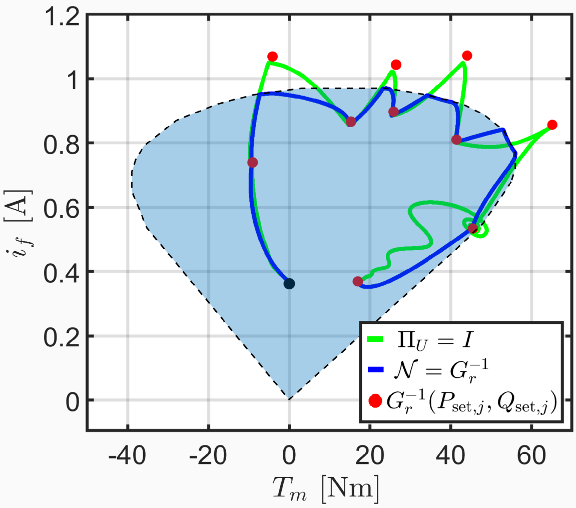
Saturating PI control loop



where $\tau_p = 0$, $k = 1$ (on the left), $k = 2$ (on the right).

Numerical Results - The Integrator State u_I and the Set U





1.2
1
0.8
0.6
0.4
0.2
0
-40
-20
0
20
40
60
i_f [A]
T_m [Nm]
$\Pi_U = I$
$\mathcal{N} = G_r^{-1}$
$G_r^{-1}(P_{\text{set},j}, Q_{\text{set},j})$

Conclusion

MIMO Sat. Integrator

Anti-Windup

Proj. Dyn. Systems

Stability Analysis

Singular Perturbation

Novel $\mathcal{N} = G_{\text{right}}^{-1}$

Application

Power Regulation of a

G.-C. Synchronverter

Publications & Awards

Publications i

- P. Lorenzetti, Z. Kustanovich, S. Shivratri, and G. Weiss.
  “The equilibrium points and stability of grid-connected synchronverters,”
  IEEE Trans. Power Systems, to appear in 2021 (available on *arXiv*).
- P. Lorenzetti and G. Weiss.
  “Saturating PI control of stable nonlinear systems using singular perturbations,”
  under review with *IEEE Trans. Aut. Control*, 2020 (available on *arXiv*).
- P. Lorenzetti and G. Weiss.
  “PI Control of stable nonlinear plants using projected dynamical systems theory,”
  about to be submitted.

P. Lorenzetti and G. Weiss.
“Integral control of stable MIMO nonlinear systems with input constraints,”
to appear in the *Proc. of the* 3^{rd} *MICNON Conference, Tokyo, September, 2021*.

P. Lorenzetti, G. Weiss and V. Natarajan.
“Integral control of stable nonlinear systems based on singular perturbations”,
IFAC-PapersOnLine, vol. 53, pp. 6157-6164, 2020.

GSC 2021 (Graduate Students in System & Control 2021) Award for the talk *"The Saturating Integrator"*, conferred by the Israeli Association for Automatic Control (IAAC) at the Technion (Haifa), in May 2021

Finalist for the Young Author Prize at the MICNON Conference 2021 with the contribution *"Integral control of stable MIMO nonlinear systems with input constraints"* (final winner to be chosen in September 2021).

Finalist for the Young Author Prize at the IFAC World Congress 2020 with the contribution *"Integral control of stable nonlinear systems based on singular perturbations"*.

Future?

Thanks for your attention!

- P. Dupuis.
  "Large deviations analysis of reflected diffusions and constrained stochastic approximation algorithms in convex sets,"
  Stochastics, vol. 21, pp. 63-96, 1987.
- P. Lorenzetti, Z. Kustanovich, S. Shivratri, and G. Weiss.
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Numerical Results - The Quantities P and Q in Time