Low order approximation of flexible structures

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$$G(s) = \frac{G_2}{s^2} + \frac{G_1}{s} + \underbrace{G_0 + G_{st}(s)}_{G_{flex}(s)}, \quad s \in \mathbb{C}. \tag{1}$$

Definition

For the system described by the transfer function (1), we define the compliance function by the following limit:

$$G_0 = \lim_{s \to 0} \left(G(s) - \frac{G_2}{s^2} - \frac{G_1}{s} \right). \tag{2}$$

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$$\omega_{assumed}(t) = \omega_2 \frac{t^2}{2} + \omega_1 t + \omega_0 + \omega_{st}(t). \tag{3}$$

Cauchy problem

Let X and U be an infinite dimensional Hilbert space. The general damped model to be considered is described as:

$$\begin{aligned} &\ddot{\omega}(t) + d\mathcal{A}_0\dot{\omega}(t) + \mathcal{A}_0\omega(t) = Bu(t), \\ &\omega(0) = 0, \quad \dot{\omega}(0) = 0, \end{aligned} \tag{4}$$

where $\mathcal{A}_0 : \mathbf{D}(\mathcal{A}_0) \subseteq X \to X$ be a self-adjoint, non-negative operator with a compact resolvent. $B : U \to X$.

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$$\dot{z}(t) = \underbrace{\begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -d\mathcal{A}_0 \end{bmatrix}}_{A_0} z(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t),$$

where $z = \left[\begin{smallmatrix} \omega \\ \dot{\omega} \end{smallmatrix}\right] \in Z := \mathbf{D}(\mathcal{A}_0^{\frac{1}{2}}) \times X$ and $\mathbf{D}(A_0) = \mathbf{D}(\mathcal{A}_0) \times \mathbf{D}(\mathcal{A}_0^{\frac{1}{2}})$.

Time domain approach

$$X = \overline{\text{span}}\{\varphi_1, \varphi_2, \varphi_3, \cdots, \varphi_{n+1}, \cdots\}.$$

By taking $u(t) = u_0, t \geq 0$ and the assumed output, the following set of equations are obtained:

$$\begin{aligned}
&\mathcal{A}_0 \omega_2 = 0, \\
&\mathcal{A}_0 \omega_1 + d\mathcal{A}_0 \omega_2 = 0, \\
&\mathcal{A}_0 \omega_0 + d\mathcal{A}_0 \omega_1 + \omega_2 = B u_0, \\
&\ddot{\omega}_{st}(t) + d\mathcal{A}_0 \dot{\omega}_{st}(t) + \mathcal{A}_0 \omega_{st}(t) = 0, \\
&\omega_{st}(0) = -\omega_0, \quad \dot{\omega}_{st}(0) = -\omega_1.
\end{aligned}$$

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&\mathcal{A}_0\omega_0 + d\mathcal{A}_0\omega_1 + \omega_2 = Bu_0,\\
&\ddot{\omega}_{st}(t) + d\mathcal{A}_0\dot{\omega}_{st}(t) + \mathcal{A}_0\omega_{st}(t) = 0,\\
&\omega_{st}(0) = -\omega_0, \quad \dot{\omega}_{st}(0) = -\omega_1.
\end{aligned}$$

$$\mathcal{A}_0\varphi_k = \lambda_k\varphi_k, \quad \text{with} \begin{cases} \lambda_k > 0, & \text{if } k \geq N+1 \\ \lambda_k = 0, & 1 \leq k \leq N \end{cases},$$

where λ_k are the eigenvalues of $\mathcal{A}_0$.

Time domain approach

Theorem

$$\omega_2 = \sum_{k=1}^{N} \alpha_k \varphi_k \quad \omega_1 = 0, \quad \omega_0 = \sum_{k=N+1}^{\infty} \beta_k \varphi_k,$$

where

$$\alpha_k = \langle \varphi_k, Bu_0 \rangle, \quad \beta_k = \frac{\langle \varphi_k, Bu_0 \rangle}{\lambda_k}.$$

Time domain approach

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- Frequency domain approach
  The transfer function from u to ω of (4) equals

$$G(s) = (s^2 + sd\mathcal{A}_0 + \mathcal{A}_0)^{-1} B.$$

since $\omega(s) = G(s)u_0$ and using the orthonormal basis of X,

$$\omega(s) = \sum_{k=1}^{\infty} \langle \varphi_k, \omega(s) \rangle \varphi_k = \sum_{k=1}^{\infty} \frac{\langle \varphi_k, Bu_0 \rangle}{s^2 + sd\lambda_k + \lambda_k} \varphi_k.$$

Frequency domain approach

By separating the calculation for $\lambda_k = 0$ and $\lambda_k > 0$, we have:

$$\omega(s) = \sum_{k=0}^{N} \langle \varphi_k, \omega \rangle \varphi_k + \sum_{k=N+1}^{\infty} \langle \varphi_k, \omega \rangle \varphi_k$$

$$= \sum_{k=0}^{N} \frac{\langle \varphi_k, Bu_0 \rangle}{s^2} \varphi_k + \sum_{k=N+1}^{\infty} \left[\frac{a_k}{s - \mu_{k,1}} + \frac{b_k}{s - \mu_{k,2}} \right] \varphi_k.$$

Hence

$$G_2 u_0 = \sum_{k=0}^{N} \langle \varphi_k, Bu_0 \rangle \phi_k, \quad G_1 = 0,$$

$$G_0 u_0 = \sum_{k=N+1}^{\infty} \frac{\langle \varphi_k, Bu_0 \rangle}{\mu_{k,1} \mu_{k,2}} \varphi_k = \sum_{k=N+1}^{\infty} \frac{\langle \varphi_k, Bu_0 \rangle}{\lambda_k} \varphi_k.$$

Boundary control problem

Let's consider:

$$\ddot{\omega}(t) + d\mathcal{A}\dot{\omega}(t) + \mathcal{A}\omega(t) = 0, \quad \omega(0) = 0, \dot{\omega}(0) = 0, \qquad (5)$$
$$\mathcal{B}_0\omega(t) = 0,$$
$$\mathcal{B}_c\omega(t) = u(t).$$

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$$\mathcal{B}_0\omega(t) = 0,$$
$$\mathcal{B}_c\omega(t) = u(t).$$

$$\mathcal{A}_0 = \mathcal{A}, \qquad \mathbf{D}(\mathcal{A}_0) = \{z \in \mathbf{D}(\mathcal{A}) \mid \mathcal{B}_0 z = 0 \text{ and } \mathcal{B}_c z = 0\}.$$

Let's define $B \in \mathcal{L}(U, X)$ such that for all $u \in U$, $Bu \in \mathbf{D}(\mathcal{A})$, the operator $\mathcal{A}B$ is an element of $\mathcal{L}(U, X)$ and,

$$\mathcal{B}_0 Bu = 0, \text{ and } \mathcal{B}_c Bu = u.$$

$$\langle z, \mathcal{A}x\rangle_X = \langle \mathcal{A}_0 z, x\rangle_X - \langle Qz, \mathcal{B}_c x\rangle_{\mathbf{U}}, \quad x \in \mathbf{D}(\mathcal{A}) \cap \ker \mathcal{B}_0, z \in \mathbf{D}(\mathcal{A}_0),$$

where Q is a linear operator from $\mathbf{D}(\mathcal{A})$ to U.

$$\ddot{v}(t) + d\mathcal{A}_0\dot{v}(t) + \mathcal{A}_0 v(t) = -B\ddot{u}(t) - d\mathcal{A}B\dot{u}(t) - \mathcal{A}Bu(t), \qquad (6)$$
$$v(0) = \epsilon - Bu_0, \dot{v}(0) = -B\dot{u}_0.$$

Time domain approach

$$
\begin{aligned}
&\mathcal{A}\omega_2 = 0,\\
&\mathcal{A}\omega_1 + d\mathcal{A}\omega_2 = 0,\\
&\mathcal{A}\omega_0 + d\mathcal{A}\omega_1 + \omega_2 = 0,\\
&\ddot{\omega}_{st}(t) + d\mathcal{A}\dot{\omega}_{st}(t) + \mathcal{A}\omega_{st}(t) = 0,\\
&\omega_{st}(0) = \epsilon - \omega_0, \quad \dot{\omega}_{st}(0) = -\omega_1.
\end{aligned}
$$

$$
\begin{cases}
\mathcal{B}_0\omega_2 = 0, \qquad \mathcal{B}_0\omega_1 = 0, \quad \mathcal{B}_0\omega_0 = 0, \quad \mathcal{B}_0\omega_{st} = 0,\\
\mathcal{B}_c\omega_2 = 0, \qquad \mathcal{B}_c\omega_1 = 0, \quad \mathcal{B}_c\omega_0 = u_0, \quad \mathcal{B}_c\omega_{st} = 0.
\end{cases}
$$

Theorem

$$
\omega_2 = \sum_{k=1}^{N} \alpha_k \varphi_k \quad \omega_1 = 0, \quad \omega_0 = \sum_{k=N+1}^{\infty} \beta_k \varphi_k,
$$

$$
\alpha_k = \langle u_0, Q\varphi_k \rangle_U, \quad \beta_k = \frac{\langle Q\varphi_k, u_0 \rangle_U}{\lambda_k}.
$$

Frequency domain approach

The transfer function from u to ω equals

$$G(s) = -(s^2 + sd\mathcal{A}_0 + \mathcal{A}_0)^{-1}(s^2 B + d\mathcal{A}Bs + \mathcal{A}B) + B.$$

$$\omega(s) = \sum_{k=0}^{N} \left[\frac{1}{s} d\langle Q\varphi_k, u_0 \rangle + \frac{1}{s^2} \langle Q\varphi_k, u_0 \rangle \right] \varphi_k +$$

$$\sum_{k=N+1}^{\infty} \left[\langle Q\varphi_k, u_0 \rangle \left(\frac{d\mu_{k,1} + 1}{\mu_{k,2} - \mu_{k,1}} \frac{1}{s - \mu_{k,1}} + \frac{d\mu_{k,2} + 1}{\mu_{k,2} - \mu_{k,1}} \frac{1}{s - \mu_{k,2}} \right) \right] \varphi_k.$$

Frequency domain approach

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$$\omega(s) = \sum_{k=0}^{N} \left[\frac{1}{s} d\langle Q\varphi_k, u_0\rangle + \frac{1}{s^2}\langle Q\varphi_k, u_0\rangle\right]\varphi_k +$$

$$\sum_{k=N+1}^{\infty} \left[\langle Q\varphi_k, u_0\rangle \left(\frac{d\mu_{k,1}+1}{\mu_{k,2}-\mu_{k,1}} \frac{1}{s-\mu_{k,1}} + \frac{d\mu_{k,2}+1}{\mu_{k,2}-\mu_{k,1}} \frac{1}{s-\mu_{k,2}}\right)\right]\varphi_k.$$

Hence

$$G_2 u_0 = \sum_{k=1}^{N} \langle Q\varphi_k, u_0\rangle \varphi_k, \quad G_1 u_0 = dG_2 u_0,$$

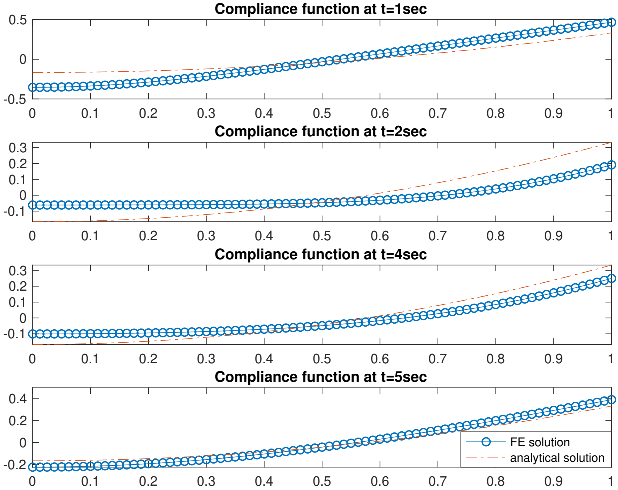
$$G_0 u_0 = \sum_{k=N+1}^{\infty} \frac{\langle Q\varphi_k, u_0\rangle}{\lambda_k}\varphi_k.$$

One-dimensional Wave equation

$$\frac{\partial^2\omega}{\partial t^2} = \frac{\partial^2\omega}{\partial x^2} + d\frac{\partial^3\omega}{\partial t\partial x^2},$$
$$\frac{\partial\omega}{\partial x}(0,t) = 0, \quad \frac{\partial\omega}{\partial x}(1,t) = u_0.$$
$$\omega(x,0^+) = 0, \quad \frac{\partial\omega}{\partial t}(x,0^+) = 0,$$

- Time domain approach
$$\begin{cases}\omega_1 = 0 \\ \mathcal{A}_0\omega_2 = 0, & \rightarrow \omega_2(x) = -u_0, \quad \omega_0(x) = -u_0\frac{x^2}{2} + C_4. \\ \mathcal{A}\omega_0 + \omega_2 = 0, \end{cases}$$
- Extra condition $\omega_{st}(x,0^+) = -\omega_0(x)$
So ω_0 lies in the stable part of system
$\int_0^1 \omega_{st}(t=0)dx = \int_0^1 -\omega_0(x)dx = 0, \rightarrow \omega_0(x) = \frac{1}{6}u_0(-1+3x^2).$

Wave equation-Time domain approach



Wave equation-Frequency domain approach

$$
\begin{aligned}
s^2 W(s,x) &= (ds+1)\frac{d^2 W(s,x)}{dx^2},\\
\frac{dW(s,0)}{dx} &= 0, \quad \frac{dW(s,1)}{dx} = u_1.
\end{aligned} \tag{7}
$$

$$
G(s) = \frac{\sqrt{1+ds}\,\cosh\left(\frac{sx}{\sqrt{1+ds}}\right)}{s\,\sinh\left(\frac{s}{\sqrt{1+ds}}\right)},
$$

For all $s \in \Omega = \left\{ s \in \mathbb{C} \mid s \neq \frac{-dk^2\pi^2 \pm k\pi\sqrt{d^2k^2\pi^2-4}}{2}, i = 1,2, k = 0,1,2,\cdots \right\}$.

Wave equation-Frequency domain approach

$$s^2 W(s,x) = (ds+1)\frac{d^2 W(s,x)}{dx^2},$$
$$\frac{dW(s,0)}{dx} = 0, \quad \frac{dW(s,1)}{dx} = u_1. \tag{7}$$

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$$G_2(x) = \lim_{s\to 0}(s^2 G(s)) = 1,$$
$$G_1(x) = \lim_{s\to 0} s(G(s) - \frac{G_2}{s^2}) = d,$$
$$G_0(x) = \lim_{s\to 0}(G(s) - \frac{G_2}{s^2} - \frac{d}{s}) = \frac{1}{6}(-1+3x^2).$$

Kirchhoff plate model

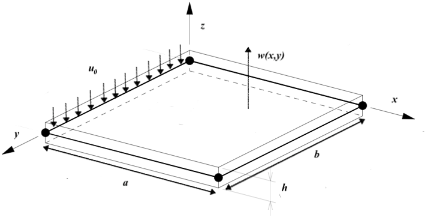
$$\rho h\ddot{\omega} + \underbrace{\kappa^T[D]\kappa}_{\mathcal{A}}\,\omega + c_d\,\underbrace{\kappa^T[D]\kappa}_{\mathcal{A}}\,\dot{\omega} = 0, \quad (x,y) \in \Omega \times [0,\infty)$$

- where

$$\mathcal{K} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x \partial y} \end{bmatrix}^T,$$

-

$$[D] = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$



Kirchhoff plate model-Time domain approach

$$\kappa^T[D]\kappa\omega_2 = 0,$$

$$\kappa^T[D]\kappa\omega_1 + d\kappa^T[D]\kappa\omega_2 = 0,$$

$$\kappa^T[D]\kappa\omega_0 + \omega_2 = 0,$$

$$\ddot{\omega}_{st}(t) + c_d\kappa^T[D]\kappa\dot{\omega}_{st}(t) + \kappa^T[D]\kappa\omega_{st}(t) = 0,$$

$$\omega_{st}(0) = \epsilon - \omega_0, \quad \dot{\omega}_{st}(0) = 0.$$

- Boundary control problem

$$\mathcal{A}_0 = \kappa^T[D]\kappa \quad \text{with } \mathbf{D}(\mathcal{A}_0) = \mathbf{D}(\mathcal{A}) \cap \ker\mathcal{B}.$$

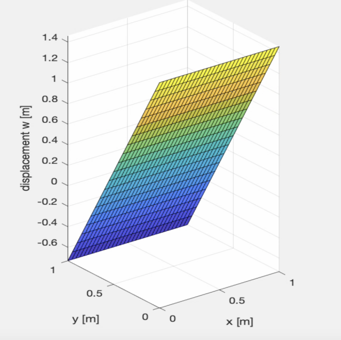
- Bu is defined by $Bu = b(x,y)u_0$ where $b(x,y)$ is a function and is contained in the domain of $\mathcal{A}$ with $\mathcal{B}Bu_0 = u_0 \quad \forall u_0 \in \mathbb{R}$ and $\mathcal{A}B = I$.

$$\mathcal{A}_0\omega_0 + \omega_2 = u_0\mathbb{1}_\Omega(x,y).$$

$$\Delta^2 B(x,y) = 1.$$

Kirchhoff plate model-numerical solution

Inertia function ω_2 [1]



[1] Numerical simulation is performed by Daniel Veldman, a graduate PhD student from Eindhoven University of technology, The Netherlands.

Kirchhoff plate model-numerical solution

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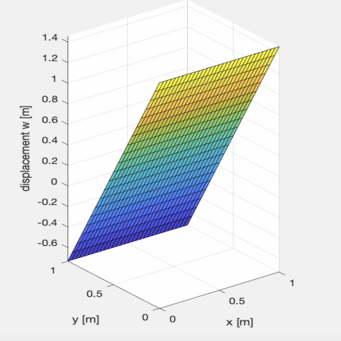
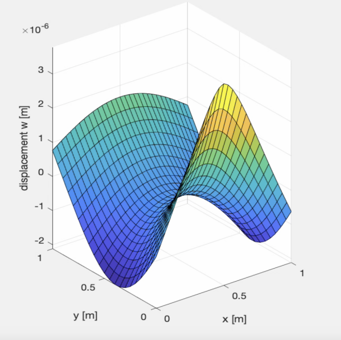


Compliance function ω_0 [1]



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Conclusion

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- A general abstract formulation for the mechanical flexible system is studied at low frequency.
- The low order approximation is investigated for the boundary control systems.
- Numerical solution for the compliance function of the Kirchhoff plate is studied.

Publications and Secondments

Publications

- **"Time domain approach toward the calculation of the compliance function of flexible motion systems "** presented at 21st IFAC World Congress, Germany, July 11-17, 2020.
- **"Linear-time-varying feedforward control for position-dependent flexible structures"** presented virtually at European Control Conference (ECC 2020), 12-15 May, 2020.
- **“Compliance feedforward of flexible structures”** presented at the 38th Benelux Meeting on Systems and Control, Lommel, Belgium, 19 – 21 March, 2019.

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Secondments

Secondments	Supervisor's name	Date
Claude Bernard University	Prof.dr. B. Maschke	M16
ASML	Dr.ir. M. Heertjes	M23

My future next journey

Thank You !