Nonlinear Port-Hamiltonian Systems

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Arjan van der Schaft

in collaboration with Bernhard Maschke, Romeo Ortega, $\cdots$

Bernoulli Institute for Mathematics, CS & AI
Jan C. Willems Center for Systems and Control
University of Groningen, the Netherlands

Most of the talk can be found in

AvdS, Dimitri Jeltsema:
Port-Hamiltonian Systems Theory: An Introductory Overview, 2014

pdf available from my home page:
www.math.rug.nl/~arjan

and in Chapters 6, 7 of

AvdS: *L_2-Gain and Passivity Techniques in Nonlinear Control*, 3rd ed. 2017

Further background:

Modeling and Control of Complex Physical Systems;
the Port-Hamiltonian Approach,
GeoPleX consortium, Springer, 2009

Introduction

- Port-Hamiltonian systems theory as systematic framework for multi-physics systems: modeling for control
- Is based on viewing energy and power as 'lingua franca' between different physical domains
- Combines classical Hamiltonian dynamics with network structure, including energy-dissipation and interaction with environment
- Unifies lumped-parameter and distributed-parameter physical systems
- Bridges the gap between modeling and control.
- Identification of underlying physical structure in the mathematical model provides powerful tools for analysis, simulation and control

Outline

The basic picture

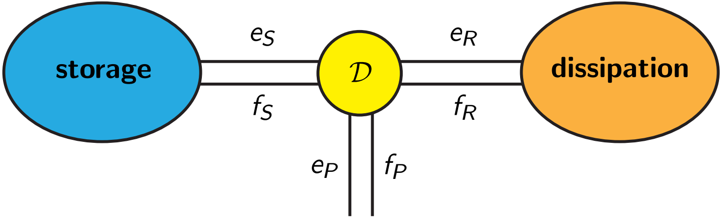


Figure: Port-Hamiltonian system

Every physical system that is modeled in this way defines a port-Hamiltonian system.

For control purposes 'any' physical system can be modeled this way.

Port-based modeling is based on viewing the physical system as interconnection of ideal basic elements, linked by energy flow.

Linking done via conjugate vector pairs of flow variables $f \in \mathbb{R}^k$ and effort variables $e \in \mathbb{R}^k$, with product $e^T f$ equal to power.

In some cases (e.g., 3D mechanical systems) $f \in \mathcal{F}$ (e.g., linear space of twists) and $e \in \mathcal{E} = \mathcal{F}^*$ (e.g., wrenches), with product defined by pairing.

Basic elements:

- (1) Energy-storing elements

$$\begin{aligned} \dot{x} &= -f \\ e &= \frac{\partial H}{\partial x}(x), \qquad H \text{ energy function} \end{aligned}$$

and hence $\frac{d}{dt}H = e^T f$.

- (2) Energy-dissipating elements:

$$R(f,e) = 0, \quad e^T f \leq 0$$

- (3) Energy-routing elements:
  - generalized transformers:

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad f_1 = Mf_2,\ e_2 = -M^T e_1$$

  - generalized gyrators:

$$f = Je, \qquad J = -J^T$$

- (4) Ideal interconnection and constraint equations:

$$e_1 = e_2 = \cdots = e_k, \quad f_1 + f_2 + \cdots + f_k = 0$$

or

$$f_1 = f_2 = \cdots = f_k, \quad e_1 + e_2 + \cdots + e_k = 0$$

$$f = 0, \quad \text{or } e = 0$$

(3) and (4) share the following two properties:

Power-conservation $e^T f = e_1 f_1 + e_2 f_2 + \cdots + e_k f_k = 0$,

and k linear and independent equations.

From energy-routing elements and interconnection equations to Dirac structures

This means energy-routing elements and interconnection and constraint equations have following two properties in common.

Described by linear equations:

$$Ff + Ee = 0, \quad f, e \in \mathbb{R}^k$$

satisfying

$$e^T f = e_1 f_1 + e_2 f_2 + \cdots + e_k f_k = 0$$

and

$$\operatorname{rank} \begin{bmatrix} F & E \end{bmatrix} = k$$

Energy-routing elements (3) and interconnection and constraint equations (4) are grouped into one geometric object: the linear space of flow and effort variables satisfying all equations, called Dirac structure.

Definition of Dirac structures

Definition

A (constant) Dirac structure is a subspace (typically $\mathcal{F} = \mathbb{R}^k = \mathcal{E}$)

$$\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$$

such that

(i) $e^T f = 0$ for all $(f, e) \in \mathcal{D}$,

(ii) $\dim \mathcal{D} = \dim \mathcal{F}$.

Example:

for any skew-symmetric map $J : \mathcal{E} \to \mathcal{F}$ its graph

$$\{(f, e) \in \mathcal{F} \times \mathcal{E} \mid f = Je\}$$

is Dirac structure.

Alternative definition; e.g., for infinite-dimensional case

Symmetrization of power $e^T f$ leads to indefinite bilinear form $\ll, \gg$ on $\mathcal{F} \times \mathcal{E}$:

$$\ll (f_a, e_a), (f_b, e_b) \gg \quad := \quad e_a^T f_b + e_b^T f_a,$$

$$(f_a, e_a), (f_b, e_b) \in \mathcal{F} \times \mathcal{E}$$

Definition

A (constant) Dirac structure is subspace

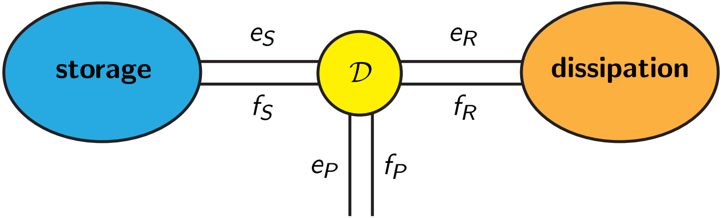
$$\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$$

such that

$$\mathcal{D} = \mathcal{D}^{\perp\!\!\perp},$$

where $\perp\!\!\perp$ denotes orthogonal companion with respect to $\ll, \gg$.

Coordinate-free definition of pH systems



Start from the Dirac structure, defined as subspace of space of all flows

$$f = (f_S, f_R, f_P)$$

and all efforts

$$e = (e_S, e_R, e_P)$$

Constitutive relations:

'Close' the energy-storing ports of $\mathcal{D}$ by relations

$$-\dot{x} = f_S, \quad \frac{\partial H}{\partial x}(x) = e_S$$

and the energy-dissipating ports by

$$R(f_R, e_R) = 0$$

This leads to the port-Hamiltonian system

$$\begin{array}{c} (-\dot{x}(t), f_R(t), f_P(t), \frac{\partial H}{\partial x}(x(t)), e_R(t), e_P(t)) \in \mathcal{D} \\ R(f_R(t), e_R(t)) = 0 \end{array} \qquad t \in \mathbb{R}$$

N.B.: in general in differential-algebraic equations (DAE) format.

Example (The ubiquitous mass-spring system)

Two energy-storage elements:

- Spring Hamiltonian $H_s(q) = \frac{1}{2}kq^2$ (potential energy)

$$\begin{aligned} \dot{q} &= -f_s && = \text{velocity} \\ e_s &= \frac{dH_s}{dq}(q) = kq && = \text{force} \end{aligned}$$

- Mass Hamiltonian $H_m(p) = \frac{1}{2m}p^2$ (kinetic energy)

$$\begin{aligned} \dot{p} &= -f_m && = \text{force} \\ e_m &= \frac{dH_m}{dp}(p) = \frac{p}{m} && = \text{velocity} \end{aligned}$$

Note the slight difference with 'classical' mechanical modeling, where one starts from identifying q as the position of mass, defining the velocity $\dot{q}$ and momentum $p = m\dot{q}$.

Example (Mass-spring system cont'd)

Dirac structure linking flows f_s, f_m, F and efforts e_s, e_m, v :

$$f_s = -e_m = -v, \quad f_m = e_s - F$$

Power-conserving since $f_s e_s + f_m e_m + vF = 0$. Yields pH system

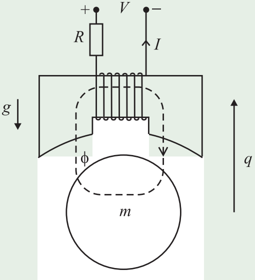
$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F$$

$$v = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}$$

with

$$H(q,p) = H_s(q) + H_m(p)$$

Example (Magnetically levitated ball)



$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -R \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p,\phi) \\ \frac{\partial H}{\partial p}(q,p,\phi) \\ \frac{\partial H}{\partial \varphi}(q,p,\phi) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V, \quad I = \frac{\partial H}{\partial \varphi}(q,p,\phi)$$

Coupling electrical/mechanical domain via Hamiltonian $H(q,p,\phi)$

$$H(q,p,\varphi) = mgq + \frac{p^2}{2m} + \frac{\varphi^2}{2L(q)}$$

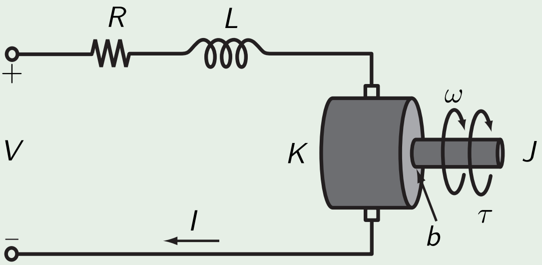
Example (Synchronous machine)

$$\begin{bmatrix} \dot{\psi}_s \\ \dot{\psi}_r \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -R_s & 0_3 & 0_{31} & 0_{31} \\ 0_3 & -R_r & 0_{31} & 0_{31} \\ 0_{13} & 0_{13} & -d & -1 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \psi_s} \\ \frac{\partial H}{\partial \psi_r} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \theta} \end{bmatrix} + \begin{bmatrix} I_3 & 0_{31} & 0_{31} \\ 0_3 & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & 0_{31} \\ 0_{13} & 0 & 1 \\ 0_{13} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_s \\ V_f \\ \tau \end{bmatrix}$$

$$\begin{bmatrix} I_s \\ I_f \\ \omega \end{bmatrix} = \begin{bmatrix} I_3 & 0_3 & 0_{31} & 0_{31} \\ 0_{13} & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} & 0 & 0 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \psi_s} \\ \frac{\partial H}{\partial \psi_r} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \theta} \end{bmatrix}, \quad R_s > 0, R_f > 0, d > 0$$

$$H(\psi_s, \psi_r, p, \theta) = \frac{1}{2} \begin{bmatrix} \psi_s^T & \psi_r^T \end{bmatrix} L^{-1}(\theta) \begin{bmatrix} \psi_s \\ \psi_r \end{bmatrix} + \frac{1}{2J} p^2$$

Example (DC motor)



6 interconnected subsystems:

• 2 energy-storing elements: inductor L with state φ (flux), and rotational inertia J with state p (angular momentum);

• 2 energy-dissipating elements: resistor R and friction b;

• gyrator K;

• voltage source V.

Example (DC motor cont'd)

Energy-storing elements (here assumed to be linear) are

$$\text{Inductor: } \begin{cases} \dot{\varphi} = -V_L \\ I = \dfrac{d}{d\varphi}\left(\dfrac{1}{2L}\varphi^2\right) = \dfrac{\varphi}{L}, \end{cases}$$

$$\text{Inertia: } \begin{cases} \dot{p} = -\tau_J \\ \omega = \dfrac{d}{dp}\left(\dfrac{1}{2J}p^2\right) = \dfrac{p}{J} \end{cases}$$

Total Hamiltonian $H(p, \phi) = \frac{1}{2L}\phi^2 + \frac{1}{2J}p^2$, and energy-dissipating relations

$$V_R = -RI, \quad \tau_b = -b\omega,$$

with $R, b > 0$, where τ_b damping torque.
Energy-routing gyrator (magnetic power into mechanical, and conversely):

$$V_K = -K\omega, \quad \tau_K = KI$$

Example (DC motor cont'd)

The subsystems are interconnected by

$$V_L + V_R + V_K + V = 0, \quad \text{while currents are equal}$$

$$\tau_J + \tau_b + \tau_K + \tau = 0, \quad \text{while angular velocities are equal}$$

Dirac structure is defined by these interconnection equations, together with equations for gyrator.

Results in port-Hamiltonian model

$$\begin{bmatrix} \dot{\varphi} \\ \dot{p} \end{bmatrix} = \left(\begin{bmatrix} 0 & -K \\ K & 0 \end{bmatrix} - \begin{bmatrix} R & 0 \\ 0 & b \end{bmatrix} \right) \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V \\ \tau \end{bmatrix},$$

$$\begin{bmatrix} I \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J} \end{bmatrix}, \qquad H(\varphi, p) = \frac{\varphi^2}{2L} + \frac{p^2}{2J}$$

Standard iso representation without algebraic constraints

In many cases the Dirac structure $\mathcal{D}$ is graph of skew-symmetric linear map

$$\begin{bmatrix} f_S \\ f_R \\ f_P \end{bmatrix} = \begin{bmatrix} -J & -G_R & -G \\ G_R^T & 0 & 0 \\ G^T & 0 & 0 \end{bmatrix} \begin{bmatrix} e_S \\ e_R \\ e_P \end{bmatrix}, \qquad J = -J^T$$

while the energy-dissipation relations are linear

$$e_R = -\bar{R} f_R, \quad \bar{R} \geq 0$$

This leads to the standard formulation

$$\begin{aligned} \dot{x} &= [J - R] \frac{\partial H}{\partial x}(x) + Gu, \quad R := G_R \bar{R} G_R^T \geq 0 \\ y &= G^T \frac{\partial H}{\partial x}(x) \end{aligned}$$

with inputs $u = f_P$ and outputs $y = e_P$.

Modulated interconnection structures

All this can be generalized to Dirac structures on manifolds $\mathcal{X}$:

$$\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$$

is a Dirac structure as before for any $x \in \mathcal{X}$.

In this case, all matrices become state-dependent, e.g.,

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + G(x)u, \quad R(x) = G_R(x)\bar{R}G_R^T(x) \geq 0 \\ y &= G^T(x)\frac{\partial H}{\partial x}(x) \end{aligned}$$

Common situation in e.g. $3D$ mechanical systems.

Mechanical systems with kinematic constraints

Consider mechanical system with n degrees of freedom. Kinematic constraints are constraints on the vector of generalized velocities $\dot{q}$:

$$A^T(q)\dot{q} = 0$$

with $A(q)$ an $n \times k$ matrix (k number of kinematic constraints).

This leads to constrained Hamiltonian equations

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda \\ 0 &= A^T(q)\frac{\partial H}{\partial p}(q,p) \end{aligned}$$

with $H(q,p)$ total energy, and $A(q)\lambda$ the constraint forces.

The resulting Dirac structure $\mathcal{D}$, modulated by $(q, p) \in T^*Q$,

is defined by the standard symplectic structure on T^*Q,

together with constraints $A^T(q)\dot{q} = 0$:

$$\mathcal{D}(q,p) = \{(f_S, e_S) \in T_{(q,p)}\mathcal{X} \times T^*_{(q,p)}\mathcal{X} \mid \exists \lambda \in \mathbb{R}^k \text{ s.t.}$$

$$f_S = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} e_S - \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda,$$

$$\begin{bmatrix} 0 & A^T(q) \end{bmatrix} e_S = 0 \quad \}$$

Energy-dissipating and external ports may be added.

Example (Rolling coin)

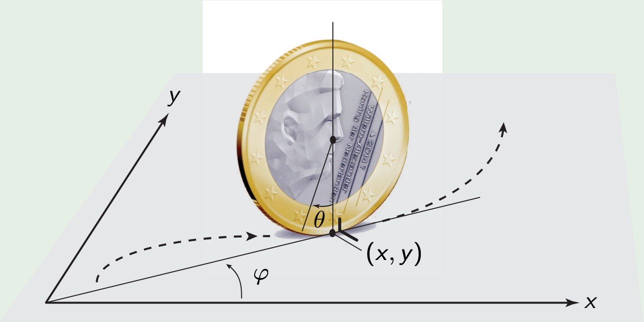


Figure: The geometry of the rolling euro

Example

Let x, y be the Cartesian coordinates of the point of contact of the coin with the plane. Furthermore, φ denotes the heading angle, and θ the angle of the coin. The rolling constraints (rolling without slipping) are (with all parameters set equal to 1)

$$\dot{x} = \dot{\theta} \cos \varphi, \quad \dot{y} = \dot{\theta} \sin \varphi$$

The total energy is (after normalization)

$$H = \frac{1}{2} p_x^2 \; + \; \frac{1}{2} p_y^2 \; + \; \frac{1}{2} p_\theta^2 \; + \; \frac{1}{2} p_\varphi^2$$

and the constraints can be rewritten in the form $A^T(q) \frac{\partial H}{\partial p}(q, p) = 0$ as

$$\begin{bmatrix} 1 & 0 & -\cos \varphi & 0 \\ 0 & 1 & -\sin \varphi & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_\theta \\ p_\varphi \end{bmatrix} = 0$$

Nonlinear energy-dissipation

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) - P(\frac{\partial H}{\partial x}(x)) + G(x)u, \quad e_S^T P(e_S) \geq 0 \\ y &= G^T(x)\frac{\partial H}{\partial x}(x) \end{aligned}$$

Example (Multi-valued nonlinear dissipation: Coulomb friction)

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} kq \\ \frac{p}{m} \end{bmatrix} - \begin{bmatrix} 0 \\ c \operatorname{sign} \frac{p}{m} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \frac{p}{m} = v$$

where sign is the multi-valued function defined by

$$\operatorname{sign} v = \begin{cases} 1 & , \quad v > 0 \\ [-1, 1] & , \quad v = 0 \\ -1 & , \quad v < 0 \end{cases}, \qquad v \operatorname{sign} v \geq 0$$

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x) \underline{-P(\frac{\partial H}{\partial x}(x)) + G(x)u}$$
$$\underline{y} = \underline{G^T(x)\frac{\partial H}{\partial x}(x)}$$

Sir William Rowan Hamilton

Addition of

- Energy-dissipating elements
- External ports $f_P = u, e_P = y$
- Algebraic constraints in case of general Dirac structure

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Outline

Mass-spring-damper systems

Associate masses to nodes, and springs and dampers to edges of a graph.

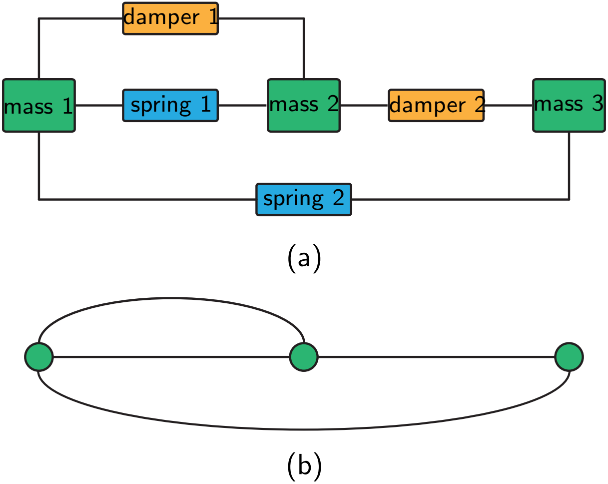


Figure: (a) Mass-spring-damper system; (b) the corresponding graph.

Mass-spring systems

For a mass-spring system with N masses and M springs in one-dimensional space $\mathbb{R}$

$$p \in \mathbb{R}^N \text{ node space}, \quad q \in \mathbb{R}^M \text{ edge space},$$

Let D be incidence matrix; then dynamics is given as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D^T \\ -D & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}$$

with total energy

$$H : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R},$$

with

$$H(q,p) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{j=1}^{M} V_j(q_j)$$

Can be directly extended to motion in $\mathbb{R}^3$, or to multi-body systems.

Mass-spring-damper systems

Part of edges correspond to springs; part to dampers.
Thus $D = \begin{bmatrix} D_s & D_d \end{bmatrix}$ with

$$D_s \text{ spring incidence matrix}, \quad D_d \text{ damper incidence matrix}$$

Dynamics of mass-spring-damper system takes the form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \left(\begin{bmatrix} 0 & D_s^T \\ -D_s & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D_d \bar{R} D_d^T \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}$$

where $\bar{R}$ is a positive diagonal matrix (in case of linear dampers).

Incidence structure defines Dirac structure (balance laws).

In electrical networks all elements are on the edges: Dirac structure determined by Kirchhoff's laws.

Chemical reaction networks: 'nonlinear mass-damper systems'.

Example: swing equation model of power network

- All voltage and current in the network are pure sinusoids with same frequency $\widehat{\omega}$ (50 Hz). Then any voltage/current signal

$$V(t) = V \sin(\widehat{\omega} t + \delta), \quad t \in \mathbb{R},$$

can be represented by its phasor

$$Ve^{j\delta}$$

- Amplitudes V_i of voltage potentials at all nodes are constant.
- All transmission lines (edges) are purely inductive.

Model the magnetic/electric part of the i-th generator/motor as a voltage source with voltage angle δ_i (and a reactance included in adjoining transmission line).
Average power ('active power') flow from node i to node j is given by

$$\Gamma_{ij} \sin(\delta_i - \delta_j)$$

with $\Gamma_{ij} = S_{ij} V_i V_j$, S_{ij} susceptance of the line from i to j.
Define phase differences across the lines

$$q_k := \delta_j - \delta_i, \quad k = 1, \cdots, m$$

Then

$$q = D^T \delta,$$

D the $n \times m$ incidence matrix of network: $n = \#$ nodes, $m \#$ lines.

It follows that vector of power flows through the lines is

$$P_{\text{network}} = -D\Gamma \,\mathrm{Sin}\, D^T \delta = -D\Gamma \,\mathrm{Sin}\, q$$

where $\mathrm{Sin} : \mathbb{R}^m \to \mathbb{R}^m$ is element-wise sin function.

Network of generators modeled by swing equations

The swing equations model the balance between mechanical and electric power as

$$M\dot{\omega} = -A\omega + P_{\mathrm{network}} + u = -A\omega - D\Gamma \,\mathrm{Sin}\, q + u$$

where $u \in \mathbb{R}^n$ is the vector of produced/consumed power at all nodes, and $A\omega$ is the vector of damping torques, with A a positive diagonal matrix.

Let ω_i be the frequency deviation with respect to $\widehat{\omega}$ of node i, then vector of phase differences $q = D^T\delta$ satisfies

$$\dot{q} = D^T\omega, \quad \omega = (\omega_1, \cdots, \omega_n)^T$$

Together, we obtain the system

$$\begin{aligned} \dot{q} &= D^T\omega \\ M\dot{\omega} &= -A\omega - D\Gamma \,\mathrm{Sin}\, q + u \end{aligned}$$

Favorite equations in control literature on power networks.

This system is naturally written into port-Hamiltonian format:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad p = M\omega$$

$$y = \frac{\partial H}{\partial p}(q,p) = \omega$$

with u vector of generated/consumed power, and Hamiltonian

$$H(q,p) = \frac{1}{2} p^T M^{-1} p - \mathbb{1}^T \Gamma \,\mathrm{Cos}\, q$$

However:

- Note that u is power, and thus the conjugated output ω is dimensionless in order that $u^T y$ is power.
- Note furthermore that ω is frequency deviation, and $p = M\omega$ is momentum deviation.
- Furthermore, $\frac{1}{2} p^T M^{-1} p$ is shifted kinetic energy, and $A\omega$ is a restoring magnetic torque; not energy dissipation.

Outline

Two key properties of port-Hamiltonian systems

Power-conservation of Dirac structure

$$e_S^T f_S + e_R^T f_R + e_P^T f_P = 0$$

implies energy-balance

$$\frac{dH}{dt}(x(t)) = \frac{\partial H}{\partial x^T}(x(t))\dot{x}(t) =$$

$$e_R^T(t) f_R(t) + e_P^T(t) f_P(t)$$

$$\leq e_P^T(t) f_P t)$$

Yields passivity of any pH system if H is bounded from below.

Crucial property for analysis and control.

Shifted passivity

In case of a constant Dirac structure, and a convex Hamiltonian, the system is also shifted passive with respect to any constant $\bar{u}$. Let e.g.,

$$0 = [J - R]\frac{\partial H}{\partial x}(\bar{x}) + G\bar{u}, \quad \bar{y} = G^T\frac{\partial H}{\partial x}(\bar{x})$$

Then

$$\begin{aligned} \dot{x} &= [J - R]\tfrac{\partial H}{\partial x}(x) + Gu, \\ y &= G^T\tfrac{\partial H}{\partial x}(x) \end{aligned}$$

can be rewritten as

$$\begin{aligned} \dot{x} &= [J - R]\tfrac{\partial \hat{H}_{\bar{x}}}{\partial x}(x) + G(u - \bar{u}), \\ y - \bar{y} &= G^T\tfrac{\partial \hat{H}_{\bar{x}}}{\partial x}(x) \end{aligned}$$

with

$$\hat{H}_{\bar{x}}(x) = H(x) - \frac{\partial H}{\partial x^T}(\bar{x})(x - \bar{x}) - H(\bar{x})$$

the shifted Hamiltonian.

Example: swing equation model of power network

Recall

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad p = M\omega$$

$$y = \frac{\partial H}{\partial p}(q,p) = \omega$$

with u vector of generated/consumed power, and Hamiltonian

$$H(q,p) = \frac{1}{2} p^T M^{-1} p - \mathbb{1}^T \Gamma \, \mathrm{Cos}\, q$$

Convex for $q \in (-\frac{\pi}{2}, \frac{\pi}{2})^n$.

Stability analysis of shifted equilibria

Let $\bar{u}$ be a constant input, yielding steady state values $(\bar{q}, \bar{p} = M\bar{\omega})$ determined by $D^T\bar{\omega} = 0$ and thus

$$\bar{\omega} = \mathbb{1}\omega_*$$

where

$$\mathbb{1}^T A \mathbb{1}\omega_* = \mathbb{1}^T \bar{u}$$

$$\text{(premultiply } 0 = -D\frac{\partial H}{\partial q}(\bar{q}, \bar{p}) - A\frac{\partial H}{\partial p}(\bar{q}, \bar{p}) + \bar{u} \text{ by } \mathbb{1}^T)$$

and furthermore

$$D\Gamma \operatorname{Sin} \bar{q} = -A\mathbb{1}\omega_* + \bar{u}$$

Note that $\omega_* = 0$ if and only if $\mathbb{1}^T\bar{u} = 0$.

Shifted Hamiltonian is

$$\widetilde{H}(q,p) := \frac{1}{2}(p-\bar{p})^T M^{-1}(p-\bar{p}) - \mathbb{1}^T \Gamma \,\mathrm{Cos}\, q + \mathbb{1}^T \Gamma \,\mathrm{Sin}\, \bar{q}\,(q-\bar{q})$$

Has a strict minimum at $(\bar{q},\bar{p})$, whenever $\bar{q} \in (-\frac{\pi}{2}, \frac{\pi}{2})^n$.

In particular, for $u = \bar{u}$ the steady state $(\bar{q},\bar{p})$ is asymptotically stable.

Similar to other dynamical distribution networks.

Port-Hamiltonian systems are compositional

The interconnection of port-Hamiltonian systems through any interconnection Dirac structure is again port-Hamiltonian:

- Total Hamiltonian H is sum of Hamiltonians of subsystems:

$$H = H_1 + \cdots + H_N$$

- Total energy-dissipating part is direct product of energy-dissipating parts of subsystems.

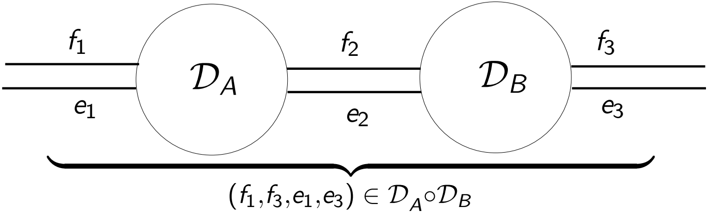
- Total Dirac structure is composition of Dirac structures of subsystems, together with interconnection Dirac structure.

Composition of Dirac structures

The composition of two Dirac structures with partially shared variables is again a Dirac structure:

$$\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2$$

$$\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{E}_2 \times \mathcal{F}_3 \times \mathcal{E}_3$$



Outline

Motivation

In many applications the system, or some of its sub-systems, is distributed-parameter.

Examples:

1. Power-converter connected to electrical machine via transmission line,
2. Hydraulic networks with fluid pipes,
3. Multi-body systems with flexible components,

etc.

Wish to combine lumped- and distributed-parameter systems into one framework.

Simplest example: transmission line

Telegrapher's equations define boundary control system

$$
\begin{aligned}
\frac{\partial Q}{\partial t}(z,t) &= -\frac{\partial}{\partial z} I(z,t) &= -\frac{\partial}{\partial z}\frac{\phi(z,t)}{L(z)} \\
\frac{\partial \phi}{\partial t}(z,t) &= -\frac{\partial}{\partial z} V(z,t) &= -\frac{\partial}{\partial z}\frac{Q(z,t)}{C(z)}
\end{aligned}
$$

$$
\begin{aligned}
f_a(t) &= V(a,t), \quad & e_a(t) &= I(a,t) \\
f_b(t) &= V(b,t), \quad & e_b(t) &= I(b,t)
\end{aligned}
$$

Stokes-Dirac structure

Define internal flows $f_S = (f_E, f_M)$ and efforts $e_S = (e_E, e_M)$:

electric flow	$f_E : [a, b] \to \mathbb{R}$
magnetic flow	$f_M : [a, b] \to \mathbb{R}$
electric effort	$e_E : [a, b] \to \mathbb{R}$
magnetic effort	$e_M : [a, b] \to \mathbb{R}$

together with boundary flows $f = (f_a, f_b)$ and efforts $e = (e_a, e_b)$.

Define infinite-dimensional subspace
$\mathcal{D} \subset (C^\infty[a, b])^2 \times (C^\infty[a, b])^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ by equations

$$\begin{bmatrix} f_E \\ f_M \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} e_E \\ e_M \end{bmatrix}$$

$$\begin{bmatrix} f_a \\ e_a \end{bmatrix} = \begin{bmatrix} e_E(a) \\ e_M(a) \end{bmatrix}, \quad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} e_E(b) \\ e_M(b) \end{bmatrix}$$

$\mathcal{D}$ is Dirac structure: $\mathcal{D} = \mathcal{D}^{\perp\!\!\perp}$

Differential operator $\frac{\partial}{\partial z}$ is skew-symmetric, as follows from integration by parts:

For any $(f_E, f_M, e_E, e_M, f_a, f_b, e_a, e_b) \in \mathcal{D}$

$$\int_a^b [e_E(z)f_E(z) + e_M(z)f_M(z)]dz - e_b f_b + e_a f_a =$$

$$\int_a^b [e_E(z)\tfrac{\partial}{\partial z}e_M(z) + e_M(z)\tfrac{\partial}{\partial z}e_E(z)]dz - e_b f_b + e_a f_a =$$

$$\int_a^b [-e_M(z)\tfrac{\partial}{\partial z}e_E(z)dz + e_M(z)\tfrac{\partial}{\partial z}e_E(z)]dz(+e_b f_b - e_a f_a) - e_b f_b + e_a f_a = 0$$

Thus $e^T f = 0$ for all $(f, e) \in \mathcal{D}$. This implies for all $(f_1, e_1), (f_2, e_2) \in \mathcal{D}$

$$0 = (e_1 + e_2)^T(f_1 + f_2) = e_1^T f_1 + e_2^T f_2 + e_1^T f_2 + e_2^T f_1 =$$

$$e_1^T f_2 + e_2^T f_1 = \ll (f_1, e_1), (f_2, e_2) \gg$$

Hence $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\perp}$.

Still need to show that $\mathcal{D}^{\perp\!\!\perp} \subset \mathcal{D}$:

Let $(\bar{f}_E, \bar{f}_M, \bar{e}_E, \bar{e}_M, \bar{f}_a, \bar{e}_a, \bar{f}_b, \bar{e}_b) \in \mathcal{D}^{\perp\!\!\perp}$, that is

$$
\begin{aligned}
0 \quad = \quad & \int_a^b [\bar{e}_E f_E + e_E \bar{f}_E + \bar{e}_M f_M + e_M \bar{f}_M] dz + \\
& -\bar{e}_b f_b - e_b \bar{f}_b + \bar{e}_a f_a + e_a \bar{f}_a
\end{aligned}
$$

for all $(f_E, f_M, e_E, e_M, f_a, e_a, f_b, e_b) \in \mathcal{D}$.
Take first $f_a = e_a = f_b = e_b = 0$. Then

$$
0 = \int_a^b [\bar{e}_E \frac{\partial}{\partial z} e_M + e_E \bar{f}_E + \bar{e}_M \frac{\partial}{\partial z} e_E + e_M \bar{f}_M] dz
$$

for all such (e_E, e_M). This implies (again integration by parts!)

$$
\bar{f}_E = \frac{\partial}{\partial z} \bar{e}_M, \quad \bar{f}_M = \frac{\partial}{\partial z} \bar{e}_E
$$

Substitution yields

$$
\begin{aligned}
0 \;=\; & \int_a^b [\bar{e}_E \frac{\partial}{\partial z} e_M + e_E \frac{\partial}{\partial z} \bar{e}_M + \bar{e}_M \frac{\partial}{\partial z} e_E + e_M \frac{\partial}{\partial z} \bar{e}_E] dz \\
& - \bar{e}_b f_b - e_b \bar{f}_b + \bar{e}_a f_a + e_a \bar{f}_a
\end{aligned}
$$

which implies

$$
\begin{aligned}
& e_E(b)\bar{e}_M(b) + e_M(b)\bar{e}_E(b) - e_E(a)\bar{e}_M(a) - e_M(a)\bar{e}_E(a) \\
& - \bar{e}_b f_b - e_b \bar{f}_b + \bar{e}_a f_a + e_a \bar{f}_a = 0
\end{aligned}
$$

for all $f_a = e_E(a), f_b = e_E(b), e_a = e_M(a), e_b = e_M(b)$.

This finally yields

$$
\bar{e}_b = \bar{e}_M(b), \quad \bar{f}_b = \bar{e}_E(b), \quad \bar{e}_a = \bar{e}_M(a), \quad \bar{f}_a = \bar{e}_E(a)
$$

Telegrapher's equations as port-Hamiltonian system

Substituting (as in the finite-dimensional case)

$$\left.\begin{array}{rcl} f_E & = & -\frac{\partial Q}{\partial t} \\ f_M & = & -\frac{\partial \varphi}{\partial t} \end{array}\right\} f_S = -\dot{x}$$

$$\left.\begin{array}{rcl} e_E & = & \frac{Q}{C} = \frac{\partial \mathcal{H}}{\partial Q}(Q,\varphi) \\ e_M & = & \frac{\varphi}{L} = \frac{\partial \mathcal{H}}{\partial \varphi}(Q,\varphi) \end{array}\right\} e_S = \frac{\partial H}{\partial x}(x)$$

with energy density

$$\mathcal{H}(Q,\varphi) = \frac{Q^2}{2C} + \frac{\varphi^2}{2L}$$

we recover the telegrapher's equations.

Extension to fluid dynamics, 3D Maxwell's equations, etc..

Interconnection of distributed-parameter pH systems and finite-dimensional pH systems

- Electrical circuits with transmission lines modeled by telegrapher's equations
- Control of boundary-control distributed-parameter systems by finite-dimensional (boundary) controllers.
- Irrigation systems: networks of fluid systems
- Dynamics of rigid bodies in fluids

Outline

Consider two heat compartments with conducting wall. The two systems, indexed by 1 and 2, exchange heat flow q given by Fourier's law

$$q = \lambda(T_1 - T_2),$$

with temperatures

$$T_i = \frac{\partial U_i}{\partial S_i}(S_i), \quad i = 1, 2,$$

with $U_1(S_1), U_2(S_2)$ internal energies of two compartments.

Leads to pseudo port-Hamiltonian system

$$\begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \end{bmatrix} = \begin{bmatrix} -\frac{q}{T_1} \\ \frac{q}{T_2} \end{bmatrix} = \begin{bmatrix} -\lambda\frac{T_1 - T_2}{T_1} \\ \lambda\frac{T_1 - T_2}{T_2} \end{bmatrix} = \begin{bmatrix} 0 & \lambda(\frac{1}{T_1} - \frac{1}{T_2}) \\ -\lambda(\frac{1}{T_1} - \frac{1}{T_2}) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial S_1} \\ \frac{\partial U}{\partial S_2} \end{bmatrix}$$

with total energy $U(S_1, S_2) := U_1(S_1) + U_1(S_2)$.

Pseudo port-Hamiltonian, since the skew-symmetric map

$$\begin{bmatrix} 0 & \lambda(\frac{1}{T_1} - \frac{1}{T_2}) \\ -\lambda(\frac{1}{T_1} - \frac{1}{T_2}) & 0 \end{bmatrix}$$

does not depend on S_1, S_2 directly, but through $T_i = \frac{\partial U_i}{\partial S_i}(S_i)$.

Therefore does not define Dirac structure on state space $\mathbb{R}^2$ with coordinates S_1, S_2: mixing of interconnection and constitutive relations.

Instead, example of the type

$$\dot{x} = J(e)e, \quad J(e) = -J^T(e), \quad e = \frac{\partial H}{\partial x}(x)$$

As a consequence

$$\dot{S}_1 + \dot{S}_2 = \frac{(T_1 - T_2)^2}{T_1 T_2} \geq 0$$

Total entropy is non-decreasing; irreversibility.

Port-Hamiltonian framework is not general enough !

Conclusions so far

- Port-based modeling of multi-physics systems: ideal energy-storage, energy-dissipation, energy-routing
- Underlying network structure defines Dirac structure
- In particular: incidence structure of graph determines Dirac structure: through and across variables
- Port-Hamiltonian modeling has been successfully applied to many situations: multi-body systems, aeronautic systems, power networks, distribution networks, chemical reaction networks, tokamak, ...
- Key properties of pH systems: passivity and compositionality
- Extension to distributed-parameter case: Stokes-Dirac structure
- Not yet enough for thermodynamics
- After the break: use for control

Some key references

- AvdS, D. Jeltsema, *Port-Hamiltonian Systems Theory: An Introductory Overview*, now publishers, 2014; see my website for pdf.
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- AvdS, B. Maschke, 'Hamiltonian formulation of distributed-parameter systems with boundary energy flow', *Journal of Geometry and Physics*, 2002.
- AvdS, B. Maschke, 'Port-Hamiltonian systems on graphs', *SIAM J. Control Optim.*, 2013.
- AvdS, B. Maschke, 'Geometry of thermodynamic processes', *Entropy*, 2018.

Outline

Introduction

Here: focus on passivity-based control of port-Hamiltonian systems,

and in particular on control by interconnection of pH systems,

(based on joint work with Romeo Ortega, Bernhard Maschke, Stefano Stramigioli, $\cdots$)

Exposition is based on parts of Chapter 7 of

AvdS, L_2-Gain and Passivity Techniques in Nonlinear Control, 3rd edition, 2017.

Recall of passivity of port-Hamiltonian systems

Power-conservation of Dirac structure

$$e_S^T f_S + e_R^T f_R + e_P^T f_P = 0$$

implies energy-balance

$$\frac{dH}{dt}(x(t)) = \frac{\partial H}{\partial x^T}(x(t))\dot{x}(t) =$$

$$e_R^T(t) f_R(t) + e_P^T(t) f_P(t)$$

$$\leq e_P^T(t) f_P t) = y^T(t) u(t)$$

Implies passivity of any pH system if H is bounded from below.

Use of passivity property for stabilization

- If $H(x) \geq 0$ (equivalent to bounded from below), with $H(x_0) = 0$, then H can be used as Lyapunov function, implying some sort of stability of x_0 for uncontrolled system.
- Furthermore, if x_0 of the uncontrolled system is only stable, then it can be sought to be asymptotically stabilized by adding artificial damping. In fact,

$$\frac{d}{dt}H \leq u^T y$$

together with additional damping $u = -y$ yields

$$\frac{d}{dt}H \leq - \parallel y \parallel^2$$

proving asymptotic stability of x_0 provided an observability condition (equivalent to LaSalle's condition for asymptotic stability) is met.

Example

Euler equations for a rigid body revolving about its center of gravity

$$\begin{aligned} I_1\dot{\omega}_1 &= [I_2 - I_3]\omega_2\omega_3 + g_1 u \\ I_2\dot{\omega}_2 &= [I_3 - I_1]\omega_1\omega_3 + g_2 u \\ I_3\dot{\omega}_3 &= [I_1 - I_2]\omega_1\omega_2 + g_3 u, \end{aligned}$$

with $\omega := (\omega_1, \omega_2, \omega_3)^T$ angular velocities around the principal axes, and $I_1, I_2, I_3 > 0$ principal moments of inertia.

For $u = 0$ the origin is an equilibrium with linearization

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} I_1^{-1} g_1 \\ I_2^{-1} g_2 \\ I_3^{-1} g_3 \end{bmatrix}$$

Hence the linearization does not say anything about stabilizability.

Stability and asymptotic stabilization by damping injection

Rewrite the system in pH form by defining angular momenta

$$p_1 = I_1\omega_1,\; p_2 = I_2\omega_2,\; p_3 = I_3\omega_3$$

and defining the Hamiltonian

$$H(p) = \frac{p_1^2}{2I_1} + \frac{p_2^2}{2I_2} + \frac{p_3^2}{2I_3}$$

System becomes

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} u, \quad y = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix}$$

Since $\dot{H} = 0$ and H has a minimum at $p = 0$ the origin is stable.

Damping injection amounts to negative output feedback

$$u = -y = -g_1 \frac{p_1}{I_1} - g_2 \frac{p_2}{I_2} - g_3 \frac{p_3}{I_3} = -g_1\omega_1 - g_2\omega_2 - g_3\omega_3,$$

yielding convergence to the largest invariant set contained in

$$\mathcal{S} := \{p \in \mathbb{R}^3 \mid \dot{H}(p) = 0\} = \{p \in \mathbb{R}^3 \mid g_1 \frac{p_1}{I_1} + g_2 \frac{p_2}{I_2} + g_3 \frac{p_3}{I_3} = 0\},$$

which is just the origin $p = 0$ if and only if

$$g_1 \neq 0, g_2 \neq 0, g_3 \neq 0,$$

in which case the origin is rendered asymptotically stable (even, globally).

Beyond control via passivity

What can we say about (asymptotic) stability of an equilibrium x_0 of the uncontrolled system if x_0 is not a minimum of the Hamiltonian ??

Recall the classical proof of stability of an equilibrium $(\omega_1^*, 0, 0) \neq (0, 0, 0)$ of the Euler equations.

The total energy $H = \frac{p_1^2}{2I_1} + \frac{p_2^2}{2I_2} + \frac{p_3^2}{2I_3}$ has minimum at $(0, 0, 0)$.
Stability of e.g. $(\omega_1^*, 0, 0)$ is shown by taking as Lyapunov function suitable combination of total energy H and angular momentum

$$C = p_1^2 + p_2^2 + p_3^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

This is a Casimir (conserved quantity independent of H) since

$$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} = 0$$

In general, for any Hamiltonian dynamics

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$$

one may search for conserved quantities C, called Casimirs, as being solutions of

$$\frac{\partial^T C}{\partial x}(x)J(x) = 0$$

Then $\frac{d}{dt}C = 0$ for *every* H, and thus also $H + C$ is a candidate Lyapunov function.

Note that minimum of $H + C$ may now be different from minimum of H.

Consider pH plant system P

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned}$$

where the desired set-point x^* is not a minimum of Hamiltonian H, and $\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$ does not possess useful Casimirs, and no shifted passivity can be used.

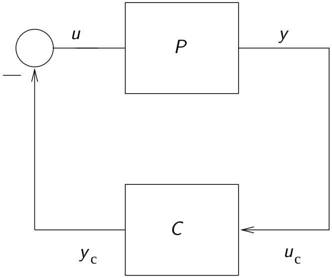
How to (asymptotically) stabilize x^* ?

Control by interconnection

Consider a controller port-Hamiltonian system

$$C: \quad \begin{array}{rcl} \dot{\xi} & = & J_c(\xi)\frac{\partial H_c}{\partial \xi}(\xi) + g_c(\xi)u_c, \quad \xi \in \mathcal{X}_c \\ y_c & = & g^T(\xi)\frac{\partial H_c}{\partial \xi}(\xi) \end{array}$$

via standard negative feedback $u = -y_c, \quad u_c = y$.



By compositionality, the closed-loop system is the pH system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} J(x) & -g(x)g_c^T(\xi) \\ g_c(\xi)g^T(x) & J_c(\xi) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \xi}(\xi) \end{bmatrix}$$

with state space $\mathcal{X} \times \mathcal{X}_c$, and total Hamiltonian $H(x) + H_c(\xi)$.

Main idea: design the controller system in such a manner that the closed-loop system has useful Casimirs $C(x, \xi)$!

This may lead to a suitable candidate Lyapunov function

$$V(x, \xi) := H(x) + H_c(\xi) + C(x, \xi)$$

with H_c still to-be-determined.

Thus we look for functions $C(x,\xi)$ satisfying

$$\begin{bmatrix} \frac{\partial^T C}{\partial x}(x,\xi) & \frac{\partial^T C}{\partial \xi}(x,\xi) \end{bmatrix} \begin{bmatrix} J(x) & -g(x)g_c^T(\xi) \\ g_c(\xi)g^T(x) & J_c(\xi) \end{bmatrix} = 0$$

such that the candidate Lyapunov function

$$V(x,\xi) := H(x) + H_c(\xi) + C(x,\xi)$$

has a minimum at (x^*,ξ^*) for some (or a set of) $\xi^* \Rightarrow$ stability.

Remark: Set of achievable closed-loop Casimirs $C(x,\xi)$ can be characterized.

In order to obtain asymptotic stability add extra damping: extend $u = -y_c, \; u_c = y$ to

$$u = -y_c - g^T(x)\frac{\partial V}{\partial x}(x,\xi), \quad u_c = y - g_c^T(x)\frac{\partial V}{\partial \xi}(x,\xi)$$

Asymptotic stability results under extra (LaSalle) conditions.

Example 1: the pendulum

Consider the mathematical pendulum with Hamiltonian

$$H(q,p) = \frac{1}{2}p^2 + (1 - \cos q)$$

actuated by torque u, with output $y = p$ (angular velocity).

Suppose we wish to stabilize the pendulum at non-zero q^* and $p^* = 0$.

Apply the nonlinear integral control

$$\begin{aligned} \dot{\xi} &= u_c = y \\ u &= -y_c = -\frac{\partial H_c}{\partial \xi}(\xi) \end{aligned}$$

which is a port-Hamiltonian controller system with $J_c = 0$.

Casimirs $C(q, p, \xi)$ are found by solving

$$\begin{bmatrix} \frac{\partial C}{\partial q} & \frac{\partial C}{\partial p} & \frac{\partial C}{\partial \xi} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = 0$$

leading to Casimirs $C(q, p, \xi) = K(q - \xi)$, and candidate Lyapunov functions

$$V(q, p, \xi) = \frac{1}{2}p^2 + (1 - \cos q) + K(q - \xi) + H_c(\xi)$$

with H_c and K to be designed. Subsequently add damping:

$$\begin{aligned} u &= -y_c - \tfrac{\partial V}{\partial p}(q, p, \xi) &&= -\tfrac{\partial H_c}{\partial \xi}(\xi) - p \\ u_c &= y - \tfrac{\partial V}{\partial \xi}(q, p, \xi) &&= p + \tfrac{\partial K}{\partial z}(q - \xi) - \tfrac{\partial H_c}{\partial \xi}(\xi) \\ \dot{\xi} &= u_c \end{aligned}$$

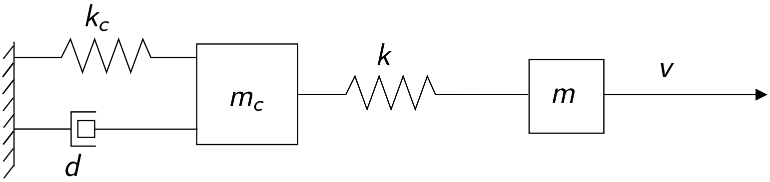


Figure: Plant mass and controller mass-spring-damper system

Consider as plant system an actuated mass m

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix}$$

with plant Hamiltonian $H(q,p) = \frac{1}{2m}p^2$ (kinetic energy).

Want to asymptotically stabilize the mass to set-point $(q^*, p^* = 0)$. Interconnect plant via

$$u = -y_c\,, u_c = y$$

to pH controller system consisting of mass m_c, two springs k_c, k, and damper d

$$\begin{bmatrix} \dot{q}_c \\ \dot{p}_c \\ \dot{\Delta q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -d & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_c}{\partial q_c} \\ \frac{\partial H_c}{\partial p_c} \\ \frac{\partial H_c}{\partial \Delta q} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_c$$

$$y_c = \frac{\partial H_c}{\partial \Delta q}$$

where q_c is extension of spring k_c, Δq extension of spring k, p_c momentum of mass m_c, $d \geq 0$ is damping constant, and u_c is external force. Controller Hamiltonian is

$$H_c\,(q_c, p_c, \Delta q) = \frac{1}{2}\frac{p_c^2}{m_c} + \frac{1}{2}k(\Delta q)^2 + \frac{1}{2}k_c q_c^2$$

Closed-loop system has Casimir functions

$$C(q, \Delta q_c, \Delta q) = q - \Delta q - q_c - \delta$$

for constant δ.

Candidate closed-loop Lyapunov function

$$V(q, \Delta q, q_c, p, p_c) := \frac{1}{2m}p^2 + \frac{1}{2m_c}p_c^2 + \frac{1}{2}k(\Delta q)^2 + \frac{1}{2}k_c q_c^2 + \gamma(q - \Delta q - q_c - \delta)^2$$

Select k, k_c, m_c, as well as δ, γ, such that V has minimum at $p = 0, q = q^*$, for some accompanying values $(\Delta q)^*, q_c^*, p_c^*$ of the controller states.

LaSalle yields asymptotic stability whenever $d > 0$.

The dissipation obstacle for generating Casimirs

Surprisingly, the presence of dissipation $R \neq 0$ may pose a problem !
C is a Casimir for pH system

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x), \quad J = J^T, R = R^T \geq 0$$

iff

$$\frac{\partial^T C}{\partial x}[J - R] = 0 \Rightarrow \frac{\partial^T C}{\partial x}[J - R]\frac{\partial C}{\partial x} = 0 \Rightarrow \frac{\partial^T C}{\partial x}R\frac{\partial C}{\partial x} = 0 \Rightarrow \frac{\partial^T C}{\partial x}R = 0$$

and thus C is a Casimir iff

$$\frac{\partial^T C}{\partial x}(x)J(x) = 0, \quad \frac{\partial^T C}{\partial x}(x)R(x) = 0$$

Similarly, if $C(x,\xi)$ is Casimir for closed-loop pH system then it must satisfy

$$\begin{bmatrix} \frac{\partial^T C}{\partial x}(x,\xi) & \frac{\partial^T C}{\partial \xi}(x,\xi) \end{bmatrix} \begin{bmatrix} R(x) & 0 \\ 0 & R_c(\xi) \end{bmatrix} = 0$$

implying by semi-positivity of $R(x)$ and $R_c(x)$

$$\frac{\partial^T C}{\partial x}(x,\xi) R(x) = 0$$

$$\frac{\partial^T C}{\partial \xi}(x,\xi) R_c(\xi) = 0$$

This is the dissipation obstacle, which implies that one cannot shape the Lyapunov function in coordinates that are directly affected by dissipation.

Physical reason for dissipation obstacle is that by using a passive controller only equilibria where no energy-dissipation takes place may be stabilized.

Remark: For shaping potential energy in mechanical systems this is not a problem since dissipation only enters in differential equations for momenta.

Example 3: Mechanical system

Mechanical system with damping and external forces u

$$\begin{aligned}
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \left(\begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D(q) \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u \\
y &= B^T(q) \frac{\partial H}{\partial p}(q,p)
\end{aligned}$$

Components of $C(x, \xi) := \xi - F(x)$ are Casimirs iff

$$J_c = 0, \qquad \frac{\partial F}{\partial q}(q,p) = g_c^T(\xi) B(q), \qquad \frac{\partial F}{\partial p}(q,p) = 0$$

Hence with $g_c(\xi) = I$ there exists solution $F(q)$ iff

$$\frac{\partial B_{il}}{\partial q_j}(q) = \frac{\partial B_{jl}}{\partial q_i}(q), \qquad i,j = 1, \dots k, \quad l = 1, \dots m$$

Nonlinear integral controller

In this example, and in many other cases, conditions for $r = n_c$ reduce to

$$\frac{\partial^T F}{\partial x}(x) J(x) \frac{\partial F}{\partial x}(x) = 0 = J_c(\xi)$$

$$\frac{\partial^T F}{\partial x}(x) J(x) = g_c(\xi) g^T(x)$$

$$R(x) \frac{\partial F}{\partial x}(x) = 0 = R_c(\xi)$$

With $g_c(\xi) = I_m$, the action of the controller pH system thus amounts to nonlinear integral action on the output y of the plant pH system:

$$\begin{aligned} u &= -\frac{\partial H_c}{\partial \xi}(\xi) + v \\ \dot{\xi} &= y + v_c \end{aligned}$$

The integral action perspective also motivates the following extension.

Consider instead of given output $y = g^T(x)\frac{\partial H}{\partial x}(x)$ any other output

$$y_A := [G(x) + P(x)]^T \frac{\partial H}{\partial x}(x) + [M(x) + S(x)]u$$

for G, P, M, S satisfying

$$g(x) = G(x) - P(x), \quad M(x) = -M^T(x), \quad \begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix} \geq 0$$

Indeed, any such alternate output satisfies

$$\frac{d}{dt}H \leq u^T y_A$$

Special choice of alternate passive output:

rewrite $\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u$ as

$$\dot{x}^T[J(x) - R(x)]^{-1}\dot{x} = \dot{x}^T\frac{\partial H}{\partial x}(x) + \dot{x}^T[J(x) - R(x)]^{-1}g(x)u$$

Since $\dot{x}^T[J(x) - R(x)]^{-1}\dot{x} \leq 0$ and $\dot{x}^T\frac{\partial H}{\partial x}(x) = \frac{d}{dt}H$ this leads to alternate output

$$\begin{aligned} y_A := \quad & g^T(x)[J(x) + R(x)]^{-1}[J(x) - R(x)]\tfrac{\partial H}{\partial x}(x) + \\ & g^T(x)[J(x) + R(x)]^{-1}g(x)u \end{aligned}$$

called the swapping the damping alternate passive output.

In particular:

Assuming $\operatorname{im} g(x) \subset \operatorname{im}[J(x) - R(x)]$ define $n \times m$ matrix $\Gamma(x)$ such that

$$[J(x) - R(x)]\Gamma(x) = g(x)$$

Then define alternate output

$$\begin{aligned} y_A : \quad = \quad & [J(x)\Gamma(x) + R(x)\Gamma(x)]^T \frac{\partial H}{\partial x}(x) \\ & + [-\Gamma^T(x)J(x)\Gamma(x) + \Gamma^T(x)R(x)\Gamma(x)]u \end{aligned}$$

Integral action $\dot{\xi} = y_A$ for arbitrary H_c leads to the following closed-loop system for $v = 0, v_c = 0$

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} J - R & -J\Gamma + R\Gamma \\ -\Gamma^T J + \Gamma^T R & \Gamma^T J\Gamma - \Gamma^T R\Gamma \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \xi}(\xi) \end{bmatrix}$$

Then

$$\begin{bmatrix} J - R & -J\Gamma + R\Gamma \\ -\Gamma^T J + \Gamma^T R & \Gamma^T J\Gamma - \Gamma^T R\Gamma \end{bmatrix} \begin{bmatrix} \Gamma \\ I_m \end{bmatrix} = 0,$$

Hence if there exist $F_1, \cdots, F_m$ such that columns of $\Gamma(x)$ satisfy

$$\Gamma_j(x) = -\frac{\partial F_j}{\partial x}(x), \quad j = 1, \cdots, m,$$

then $\xi_j - F_j(x)$, $j = 1, \cdots, m$, are Casimirs of the closed-loop system.

Example

Consider an RLC-circuit with voltage source u, where the capacitor is in parallel with the resistor. Dynamics

$$\begin{bmatrix} \dot{Q} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R} & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{Q}{C} \\ \frac{\phi}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Suppose we want to stabilize the system at some non-zero set-point $(Q^*, \phi^*) = (C\bar{u}, \frac{L}{R}\bar{u})$ for $\bar{u} \neq 0$.

Example

Integral action of natural passive output $y = \frac{\phi}{L}$ (current through voltage source) does not help in creating Casimirs. Instead consider solution $\Gamma^T = \begin{bmatrix} 1 & \frac{1}{R} \end{bmatrix}$, and resulting alternate passive output

$$y_A = \frac{2}{R}\frac{Q}{C} - \frac{\phi}{L} + \frac{1}{R}u$$

Integral action yields Casimir $Q + \frac{1}{R}\phi - \xi$ for closed-loop system, resulting in candidate Lyapunov function

$$V(Q, \phi, \xi) = \frac{1}{2C}Q^2 + \frac{1}{2L}\phi^2 + H_c(\xi) + \Phi(Q + \frac{1}{R}\phi - \xi)$$

H_c and Φ can be found s.t. V has minimum at (Q^*, ϕ^*, ξ^*) for some ξ^*.
In series RLC circuit integral action of natural output suffices, resulting in controller system that emulates an extra capacitor.
Main difference is that in parallel RLC circuit there is energy dissipation at equilibrium whenever $\bar{u} \neq 0$, in contrast to series case.

State feedback perspective

Suppose there exists a solution F to Casimir equations with $r = n_c$, in which case all controller states ξ are related to the plant states x. Then for any choice of vector of constants $\lambda = (\lambda_1, \cdots, \lambda_{n_c})$

$$L_\lambda := \{(x, \xi) \mid \xi_i = F_i(x) + \lambda_i, \; i = 1, \ldots, n_c\}$$

is an invariant manifold of the closed-loop system for $v = 0, v_c = 0$. Furthermore, dynamics restricted to L_λ is given as

$$\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) - g(x) g_c^T (F(x) + \lambda) \frac{\partial H_c}{\partial \xi}(F(x) + \lambda)$$

In fact

$$\dot{x} = [J(x) - R(x)] \frac{\partial H_s}{\partial x}(x)$$

with

$$H_s(x) := H(x) + H_\lambda(x), \quad H_\lambda(x) := H_c(F(x) + \lambda),$$

defining pH system with same $J(x)$ and $R(x)$, but shaped Hamiltonian H_s.

Alternatively, the dynamics could have been obtained directly by applying to plant pH system state feedback $u = \alpha_\lambda(x)$ such that

$$g(x)\alpha_\lambda(x) = [J(x) - R(x)] \frac{\partial H_\lambda}{\partial x}(x)$$

In fact

$$\alpha_\lambda(x) = -g_c^T(F(x) + \lambda)\frac{\partial H_c}{\partial \xi}(F(x) + \lambda)$$

Since Casimirs are defined up to a constant we can also leave out dependence on λ and simply consider

$$\alpha(x) := -g_c^T(F(x))\frac{\partial H_c}{\partial \xi}(F(x))$$

for any solution F.

Find $u = \alpha(x)$ and $h(x)$ satisfying

$$[J(x) - R(x)]\, h(x) = g(x)\alpha(x)$$

such that

$$\begin{array}{rrcll} (i) & \frac{\partial h_i}{\partial x_j}(x) & = & \frac{\partial h_j}{\partial x_i}(x), & \qquad i,j = 1,\ldots,n \\ (ii) & h(x^*) & = & -\frac{\partial H}{\partial x}(x^*) & \\ (iii) & \frac{\partial h}{\partial x}(x^*) & > & -\frac{\partial^2 H}{\partial x^2}(x^*) & \end{array}$$

with $\frac{\partial h}{\partial x}(x)$ the $n \times n$ matrix with i-th column given by $\frac{\partial h_i}{\partial x}(x)$, and $\frac{\partial^2 H}{\partial x^2}(x^*)$ the Hessian matrix of H at x^*.
Then x^* is stable equilibrium of closed-loop system

$$\dot{x} = [J(x) - R(x)]\, \frac{\partial H_d}{\partial x}(x)$$

where $H_d(x) := H(x) + H_a(x)$, with

$$h(x) = \frac{\partial H_a}{\partial x}(x)$$

Example

Hamiltonian H of rolling coin does not have strict minimum at the desired equilibrium $x = y = \theta = \phi = 0, \quad p_1 = p_2 = 0$, since the potential energy is zero. Consider

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ -\cos\phi & -\sin\phi & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial P_a}{\partial x} \\ \frac{\partial P_a}{\partial y} \\ \frac{\partial P_a}{\partial \theta} \\ \frac{\partial P_a}{\partial \phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

with P_a and α_1, α_2 functions of x, y, θ, ϕ.

Taking $P_a(x, y, \theta, \phi) = \frac{1}{2}(x^2 + y^2 + \theta^2 + \phi^2)$ leads to state feedback

$$\begin{aligned} u_1 &= -x\cos\phi - y\sin\phi - \theta + v_1 \\ u_2 &= -\phi + v_2 \end{aligned}$$

Elimination of $\alpha(x)$

Conditions

$$[J(x) - R(x)]\, h(x) = g(x)\alpha(x)$$

can be simplified to conditions on $h(x)$ only:

Let $g(x)$ be full column rank for every $x \in \mathcal{X}$. Denote by $g^{\perp}(x)$ a matrix of maximal rank such that $g^{\perp}(x)g(x) = 0$. Let $h(x), \alpha(x)$ be solution. Then $h(x)$ is solution to

$$g^{\perp}(x)[J(x) - R(x)]h(x) = 0$$

Conversely, if $h(x)$ is a solution to the latter then there exists $\alpha(x)$ such that $h(x), \alpha(x)$ is solution to the first. In fact,

$$\alpha(x) = \left(g^T(x)g(x)\right)^{-1} g^T(x)\, [J(x) - R(x)]\, h(x)$$

Outline

1. Port-Hamiltonian systems
2. Port-Hamiltonian formulation of network dynamics
3. Properties of port-Hamiltonian systems
4. Distributed-parameter port-Hamiltonian systems
5. Including thermodynamics ?
6. Passivity-based control of port-Hamiltonian systems
7. IDA Passivity-based control
8. New control paradigms emerging

Interconnection-Damping Assignment (IDA)-PBC control

Further possibility to generate candidate Lyapunov functions H_d is to look for state feedbacks $u = \hat{u}_{IDA}(x)$ such that

$$[J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u_{IDA}(x) = [J_d(x) - R_d(x)]\frac{\partial H_d}{\partial x}(x)$$

where J_d and R_d are newly assigned interconnection and damping structures.

As before this reduces to finding H_d, J_d, R_d such that

$$g^{\perp}(x)\,[J(x) - R(x)]\,\frac{\partial H}{\partial x}(x) = g^{\perp}(x)\,[J_d(x) - R_d(x)]\,\frac{\partial H_d}{\partial x}(x)$$

Interesting theory especially for mechanical systems.

Much more to be said; see e.g. work of Romeo Ortega and co-workers.

Outline

Energy flow control

Consider two port-Hamiltonian systems

$$
\begin{aligned}
\dot{x}_i &= J_i(x_i)\frac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i)u_i \\
y_i &= g_i^T(x_i)\frac{\partial H_i}{\partial x_i}(x_i), \qquad i = 1, 2
\end{aligned}
$$

Suppose we want to transfer energy from system 1 to system 2, while keeping total energy $H_1 + H_2$ constant.

Use output feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -y_1 y_2^T \\ y_2 y_1^T & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

It follows that the closed-loop system is energy-preserving. However, for the individual energies

$$\frac{d}{dt} H_1 = -y_1^T y_1 y_2^T y_2 = -||y_1||^2 ||y_2||^2 \leq 0$$

implying that H_1 is decreasing as long as $||y_1||$ and $||y_2||$ are different from 0. On the other hand,

$$\frac{d}{dt} H_2 = y_2^T y_2 y_1^T y_1 = ||y_2||^2 ||y_1||^2 \geq 0$$

implying that H_2 is increasing at the same rate.

Has been successfully applied to energy-efficient path-following control of mechanical systems (Duindam & Stramigioli).

NB: results in pseudo-Poisson structure of closed-loop system; similar to heat conduction example before.

Impedance control

Consider a system with two (not necessarily distinct) ports

$$\begin{aligned}
\dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u + k(x)f, \quad x \in \mathcal{X} \\
y &= g^T(x)\frac{\partial H}{\partial x}(x), \quad u, y \in \mathbb{R}^m \\
e &= k^T(x)\frac{\partial H}{\partial x}(x), \quad f, e \in \mathbb{R}^m
\end{aligned}$$

Relation between f and e is called the 'impedance' of (f, e)-port.

In Impedance Control (Hogan) one tries to shape this impedance by using the control port (u, y).

Typical application: the (f, e)-port corresponds to end-tip of robotic manipulator, while the (u, y)-port corresponds to actuation.

Basic question: what are achievable impedances of the (f, e)-port ?, and how to shape by control the impedance to a desired one ?

Dirac structures depending on u, and variable transmission

(see also Folkertsma & Stramigioli: Energy in Robotics)

Main idea: control the system by routing the power flows in desirable manner by modulating $\mathcal{D}(u)$, based on information about state variables.

Aim: energy-efficient control with higher performance than 'ordinary' passive control; achieving control aims without adding damping.

In power converters this is a natural scenario: Dirac structure (determined by Kirchhoff's laws) depends on (to-be-controlled) duty-ratios of switches.

In mechanical systems it corresponds to variable transmission.

Variable stiffness control

A variable stiffness controller is defined by a (virtual) linear spring with energy

$$H(q) = \frac{1}{2}kq^2,$$

where we regard stifness k as extra state variable whose value may change over time.

This leads to consideration of pH system

$$\begin{bmatrix} \dot{q} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} kq \\ \frac{1}{2}q^2 \end{bmatrix}$$

The port (u_1, y_1) corresponds to interaction with the environment.

The port (u_2, y_2) defines a control port, regulating the stiffness k based on the output $y_2 = \frac{1}{2}q^2$, possibly modulated by information about other variables in the total system.

Conclusions and Outlook

Much more work on pH systems has been done:

- Switching pH systems (e.g., electrical circuits with diods and switches; robotic walking)
- Relation with L_2-gain theory via scattering
- Pseudo-gradient formulations (Brayton-Moser)
- Spatial discretization of distributed-parameter pH systems
- Time-discretization for simulation
- Structure-preserving model reduction of pH systems
- Applications to power systems and chemical reaction networks

Very much open:

Port-Hamiltonian identification theory and data-driven control.

Epilogue

Control by interconnection of pH systems regards controller system as another pH system; either physical or emulating a physical system (e.g., interpretation of PI-controller as addition of damper and spring.)

Prevailing paradigm: controller system is 'physical' system interacting with the plant system via energy flow.

Advantages: stable (interaction with environment!) and often robust, physically interpretable.

Disadvantages: control by interconnection (not IDA-PBC) is often collocated control; performance may not be optimal.

Question: How about information flow? How about the paradigm of control as 'information gathering, processing and applying' ?
Observer design ?

Can thermodynamics help in uniting both paradigms ?