Ingham's inequality

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An Introduction to the Controllability of Partial Differential Equations

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Inverse Spectral Theory

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In this lecture we present two inequalities which have been successfully used in the study of many 1-D control problems and, more precisely, to prove observation inequalities. They generalize the classical Parseval's equality for orthogonal sequences. Variants of these inequalities were studied in the works of Paley and Wiener at the beginning of the past century (see [PW]).
The main inequality was proved by Ingham (see [I]) who gave a beautiful and elementary proof (see Theorems 1 and 2 below). Since then, many generalizations have been given (see, for instance, [BS], [HA], [BKL] and [JM]).

Theorem

(Ingham [I]) Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ be such that

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{1}$$

For any real T with

$$T > \pi/\gamma \tag{2}$$

there exists a positive constant $C_1 = C_1(T, \gamma) > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$C_1 \sum_{n\in\mathbb{Z}} | \, a_n \, |^2 \leq \int_{-T}^{T} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt. \tag{3}$$

Proof

We first reduce the problem to the case $T = \pi$ and $\gamma > 1$. Indeed, if T and γ are such that $T\gamma > \pi$, then

$$\int_{-T}^{T} \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt \quad = \quad \frac{T}{\pi} \int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\frac{T\lambda_n}{\pi}s} \right|^2 ds$$

$$= \frac{T}{\pi} \int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\mu_n s} \right|^2 ds$$

where $\mu_n = T\lambda_n/\pi$. It follows that
$\mu_{n+1} - \mu_n = T\left(\lambda_{n+1} - \lambda_n\right)/\pi \geq \gamma_1 := T\gamma/\pi > 1.$
We prove now that there exists $C_1' > 0$ such that

$$C_1' \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \leq \int_{-\pi}^{\pi} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\mu_n t} \right|^2 dt.$$

Define the function

$$h:\mathbb{R}\to\mathbb{R},\ h(t)=\begin{cases}\cos(t/2) & \text{if} \quad |\, t \,|\leq \pi\\ 0 & \text{if} \quad |\, t \,|> \pi\end{cases}$$

and let us compute its Fourier transform $K(\varphi)$,

$$K(\varphi)=\int_{-\pi}^{\pi}h(t)e^{it\varphi}dt=\int_{-\infty}^{\infty}h(t)e^{it\varphi}dt=\frac{4\cos\pi\varphi}{1-4\varphi^2}.$$

On the other hand, since $0\leq h(t)\leq 1$ for any $t\in[-\pi,\pi]$, we have that

$$\int_{-\pi}^{\pi}\left|\sum_n a_n e^{i\mu_n t}\right|^2 dt \geq \int_{-\pi}^{\pi}h(t)\left|\sum_n a_n e^{i\mu_n t}\right|^2 dt=\sum_{n,m}a_n\bar{a}_m K(\mu_n-\mu_m)=$$

$$=K(0)\sum_n |\, a_n \,|^2+\sum_{n\neq m}a_n\bar{a}_m K(\mu_n-\mu_m)\geq$$

$$\geq 4\sum_n |\, a_n \,|^2-\frac{1}{2}\sum_{n\neq m}\left(|\, a_n \,|^2+|\, a_m \,|^2\right)|\, K(\mu_n-\mu_m)\,|=$$

$$=4\sum_n |\, a_n \,|^2-\sum_n |\, a_n \,|^2\sum_{m\neq n}|\, K(\mu_n-\mu_m)\,|.$$

Remark that

$$\sum_{m\neq n} \mid K(\mu_n-\mu_m) \mid \leq \sum_{m\neq n} \frac{4}{4\mid \mu_n-\mu_m\mid^2-1} \leq \sum_{m\neq n}\frac{4}{4\gamma_1^2\mid n-m\mid^2-1}=$$

$$=8\sum_{r\geq 1}\frac{1}{4\gamma_1^2 r^2-1}\leq \frac{8}{\gamma_1^2}\sum_{r\geq 1}\frac{1}{4r^2-1}=\frac{8}{\gamma_1^2}\frac{1}{2}\sum_{r\geq 1}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)=\frac{4}{\gamma_1^2}.$$

Hence,

$$\int_{-\pi}^{\pi}\left|\sum_n a_n e^{i\mu_n t}\right|^2 dt \geq \left(4-\frac{4}{\gamma_1^2}\right)\sum_n \mid a_n\mid^2 .$$

If we take

$$C_1=\frac{T}{\pi}\left(4-\frac{4}{\gamma_1^2}\right)=\frac{4\pi}{T}\left(T^2-\frac{\pi^2}{\gamma^2}\right)$$

the proof is concluded.

Theorem

Let $(\lambda_n)_{n\in\mathbb{Z}}$ *be a sequence of real numbers and* $\gamma > 0$ *be such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{4}$$

For any $T > 0$ *there exists a positive constant* $C_2 = C_2(T,\gamma) > 0$ *such that, for any finite sequence* $(a_n)_{n\in\mathbb{Z}}$*,*

$$\int_{-T}^{T} \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt \leq C_2 \sum_n \mid a_n \mid^2 . \tag{5}$$

Proof

Let us first consider the case $T\gamma \geq \pi/2$. As in the proof of the previous theorem, we can reduce the problem to $T = \pi/2$ and $\gamma \geq 1$. Indeed,

$$\int_{-T}^{T} \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt \; = \; \frac{2T}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\mu_n s} \right|^2 ds$$

where $\mu_n = 2T\lambda_n/\pi$. It follows that
$\mu_{n+1} - \mu_n = 2T\left(\lambda_{n+1} - \lambda_n\right)/\pi \geq \gamma_1 := 2T\gamma/\pi \geq 1$.
Let h be the function introduced in the proof of Theorem 1. Since $\sqrt{2}/2 \leq h(t) \leq 1$ for any $t \in [-\pi/2, \pi/2]$ we obtain that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \leq 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(t) \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \leq$$

$$\leq 2 \int_{-\pi}^{\pi} h(t) \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt = 2 \sum_{n,m} a_n \bar{a}_m K\left(\mu_n - \mu_m\right) =$$

$$= 8 \sum_{n} | a_n |^2 + 2 \sum_{n \neq m} a_n \bar{a}_m K(\mu_n - \mu_m) \leq$$

$$\leq 8 \sum_{n} | a_n |^2 + \sum_{n \neq m} \left(| a_n |^2 + | a_m |^2 \right) | K(\mu_n - \mu_m) | .$$

As in the proof of Theorem 1 we obtain that

$$\sum_{m \neq n} | K(\mu_n - \mu_m) | \leq \frac{4}{\gamma_1^2}.$$

Hence,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \leq 8 \sum_n | a_n |^2 + \frac{8}{\gamma_1^2} \sum_n | a_n |^2 \leq 8 \left(1 + \frac{1}{\gamma_1^2} \right) \sum_n | a_n |^2$$

and (5) follows with $C_2 = 8 \left(4T^2/(\pi^2) + 1/\gamma^2 \right)$.

When $T\gamma < \pi/2$ we have that

$$\int_{-T}^{T} \left| \sum a_n e^{i\lambda_n t} \right|^2 dt = \frac{1}{\gamma} \int_{-T\gamma}^{T\gamma} \left| \sum a_n e^{i\frac{\lambda_n}{\gamma} s} \right|^2 ds \leq \frac{1}{\gamma} \int_{-\pi/2}^{\pi/2} \left| \sum a_n e^{i\frac{\lambda_n}{\gamma} s} \right|^2 ds$$

Since $\lambda_{n+1}/\gamma - \lambda_n/\gamma \geq 1$ from the analysis of the previous case we obtain that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\frac{\lambda_n}{\gamma}s} \right|^2 ds \leq 16 \sum_n \mid a_n \mid^2 .$$

Hence, (5) is proved with

$$C_2 = 8 \max \left\{ \left(\frac{4T^2}{\pi^2} + \frac{1}{\gamma^2} \right), \ \frac{2}{\gamma} \right\}$$

and the proof concludes.

Remarks

- Inequality (5) holds for all $T > 0$. On the contrary, inequality (3) requires the length T of the time interval to be sufficiently large. Note that, when the distance between two consecutive exponents λ_n, the gap, becomes small the value of T must increase proportionally.
- In the first inequality (3) T depends on the minimum γ of the distances between every two consecutive exponents (gap). However, as we shall see in the next theorem, only the asymptotic distance as $n \to \infty$ between consecutive exponents really matters to determine the minimal control time T. Note also that the constant C_1 in (3) degenerates when T goes to π/γ.

- In the critical case $T = \pi/\gamma$ inequality (3) may hold or not, depending on the particular family of exponential functions. For instance, if $\lambda_n = n$ for all $n \in \mathbb{Z}$, (3) is verified for $T = \pi$. This may be seen immediately by using the orthogonality property of the complex exponentials in $(-\pi, \pi)$. Nevertheless, if $\lambda_n = n - 1/4$ and $\lambda_{-n} = -\lambda_n$ for all $n > 0$, (3) fails for $T = \pi$ (see, [I] or [Y]).

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As we have said before, the length $2T$ of the time interval in (3) does not depend on the smallest distance between two consecutive exponents but on the asymptotic gap defined by

$$\varliminf_{|n|\to\infty} \mid \lambda_{n+1} - \lambda_n \mid = \gamma_\infty. \tag{6}$$

An induction argument due to A. Haraux (see [H]) allows to give a result similar to Theorem 1 above in which condition (1) for γ is replaced by a similar one for γ_∞.

Theorem

Let $(\lambda_n)_{n\in\mathbb{Z}}$ be an increasing sequence of real numbers such that $\lambda_{n+1} - \lambda_n \geq \gamma > 0$ for any $n \in \mathbb{Z}$ and let $\gamma_\infty > 0$ be given by (6). For any real T with

$$T > \pi/\gamma_\infty \tag{7}$$

there exist two positive constants $C_1, C_2 > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$C_1 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \leq \int_{-T}^{T} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt \leq C_2 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 . \tag{8}$$

When $\gamma_\infty = \gamma$, the sequence of Theorem 3 satisfies $\lambda_{n+1} - \lambda_n \geq \gamma_\infty > 0$ for all $n \in \mathbb{Z}$ and we can then apply Theorems 1 and 2. However, in general, $\gamma_\infty < \gamma$ and Theorem 3 gives a sharper bound on the minimal time T needed for (8) to hold.

Note that the existence of C_1 and C_2 in (8) is a consequence of Kahane's theorem (see [K]). However, if our purpose were to have an explicit estimate of C_1 or C_2 in terms of γ, γ_∞ then we would need to use the constructive argument below. It is important to note that these estimates depend strongly also on the number of eigenfrequencies λ that fail to fulfill the gap condition with the asymptotic gap γ_∞.

Proof of Theorem 3:

The second inequality from (8) follows immediately by using Theorem 2. To prove the first inequality (8) we follow the induction argument due to Haraux [H].
Note that for any $\varepsilon_1 > 0$, there exists $N = N(\varepsilon_1) \in \mathbb{N}^*$ such that

$$|\lambda_{n+1} - \lambda_n| \geq \gamma_\infty - \varepsilon_1 \text{ for any } |n| > N. \tag{9}$$

We begin with the function $f_0(t) = \sum_{|n|>N} a_n e^{i\lambda_n t}$ and we add the missing exponentials one by one. From (9) we deduce that Theorems 1 and 2 may be applied to the family $\left(e^{i\lambda_n t}\right)_{|n|>N}$ for any $T > \pi/(\gamma_\infty - \varepsilon_1)$

$$C_1 \sum_{n>N} \mid a_n \mid^2 \leq \int_{-T}^{T} \mid f_0(t) \mid^2 dt \leq C_2 \sum_{n>N} \mid a_n \mid^2 . \tag{10}$$

Let now $f_1(t) = f_0 + a_N e^{i\,\lambda_N t} = \sum_{|n|>N} a_n e^{i\lambda_n t} + a_N e^{i\,\lambda_N t}$. Without loss of generality we may suppose that $\lambda_N = 0$ (since we can consider the function $f_1(t)e^{-i\lambda_N t}$ instead of $f_1(t)$).

Let $\varepsilon > 0$ be such that $T' = T - \varepsilon > \pi/\gamma_\infty$. We have $\Big[$

$$\int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta = \sum_{n>N} a_n \left(\frac{e^{i\lambda_n\varepsilon} - 1}{i\lambda_n} - \varepsilon \right) e^{i\lambda_n t}, \quad \forall t \in [0, T'].\Big]$$

Applying now (10) to the function $h(t) = \displaystyle\int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta$ we obtain that:

$$C_1 \sum_{n>N} \left| \frac{e^{i\lambda_n\varepsilon} - 1}{i\lambda_n} - \varepsilon \right|^2 |a_n|^2 \leq \int_{-T'}^{T'} \left| \int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta \right|^2 dt. \tag{11}$$

Moreover,: $\Big[\ \left| e^{i\lambda_n\varepsilon} - 1 - i\lambda_n\varepsilon \right|^2 = |\cos(\lambda_n\varepsilon) - 1|^2 + |\sin(\lambda_n\varepsilon) - \lambda_n\varepsilon|^2 =$

$$\Big]\Big[= 4\sin^4\left(\frac{\lambda_n\varepsilon}{2}\right) + (\sin(\lambda_n\varepsilon) - \lambda_n\varepsilon)^2 \geq \begin{cases} 4\left(\frac{\lambda_n\varepsilon}{\pi}\right)^4, & \text{if } |\lambda_n|\varepsilon \leq \pi \\ (\lambda_n\varepsilon)^2, & \text{if } |\lambda_n|\varepsilon > \pi. \end{cases} \Big]$$

Finally, taking into account that $|\lambda_n| \geq \gamma$, we obtain that,

$$\left| \frac{e^{i\lambda_n \varepsilon} - 1}{i\lambda_n} - \varepsilon \right|^2 \geq c\varepsilon^2.$$

We return now to (11) and we get that:

$$\varepsilon^2 C_1 \sum_{n>N} |a_n|^2 \leq \int_{-T'}^{T'} \left| \int_0^{\varepsilon} \left(f_1(t+\eta) - f_1(t) \right) d\eta \right|^2 dt. \tag{12}$$

On the other hand

$$\int_{-T'}^{T'} \left| \int_0^{\varepsilon} \left(f_1(t+\eta) - f_1(t) \right) d\eta \right|^2 dt \leq \int_{-T'}^{T'} \varepsilon \int_0^{\varepsilon} |f_1(t+\eta) - f_1(t)|^2 \, d\eta dt \leq$$

$$\leq 2\varepsilon \int_{-T'}^{T'} \int_0^{\varepsilon} \left(|f_1(t+\eta)|^2 + |f_1(t)|^2 \right) d\eta dt \leq 2\varepsilon^2 \int_{-T'}^{T} |f_1(t)|^2 \, dt +$$

$$+2\varepsilon \int_0^{\varepsilon} \int_{-T'}^{T'} |f_1(t+\eta)|^2 \, dt d\eta = 2\varepsilon^2 \int_{-T'}^{T} |f_1(t)|^2 \, dt + 2\varepsilon \int_0^{\varepsilon} \int_{-T'+\eta}^{T'+\eta} |f_1(s)|^2 \, ds$$

$$\leq 2\varepsilon^2 \int_{-T}^{T} |f_1(t)|^2 \, dt + 2\varepsilon \int_0^{\varepsilon} \int_{-T}^{T} |f_1(s)|^2 \, ds d\eta \leq 4\varepsilon^2 \int_{-T}^{T} |f_1(t)|^2 \, dt.$$

From (12) it follows that

$$C_1 \sum_{n>N} |a_n|^2 \leq \int_{-T}^{T} |f_1(t)|^2 \, dt. \tag{13}$$

On the other hand

$$|a_N|^2 = \left| f_1(t) - \sum_{n>N} a_n e^{i\lambda_n t} \right|^2 = \frac{1}{2T} \int_{-T}^{T} \left| f_1(t) - \sum_{n>N} a_n e^{i\lambda_n t} \right|^2 dt \leq$$

$$\leq \frac{1}{T} \left(\int_{-T}^{T} |f_1(t)|^2 \, dt + \int_{-T}^{T} \left| \sum_{n>N} a_n e^{i\lambda_n t} \right|^2 \right) dt \leq$$

$$\leq \frac{1}{T} \left(\int_{-T}^{T} |f_1(t)|^2 \, dt + C_2^0 \sum_{n>N} |a_n|^2 \right) \leq$$

$$\leq \frac{1}{T} \left(1 + \frac{C_2}{C_1} \right) \int_{-T}^{T} |f_1(t)|^2 \, dt.$$

From (13) we get that

$$C_1 \sum_{n \geq N} |a_n|^2 \leq \int_{-T}^{T} |f_1(t)|^2 \, dt.$$

Repeating this argument we may add all the terms $a_n e^{i\lambda_n t}$, $|n| \leq N$ and we obtain the desired inequalities.