1-Waves: Propagation, Control & Numerics

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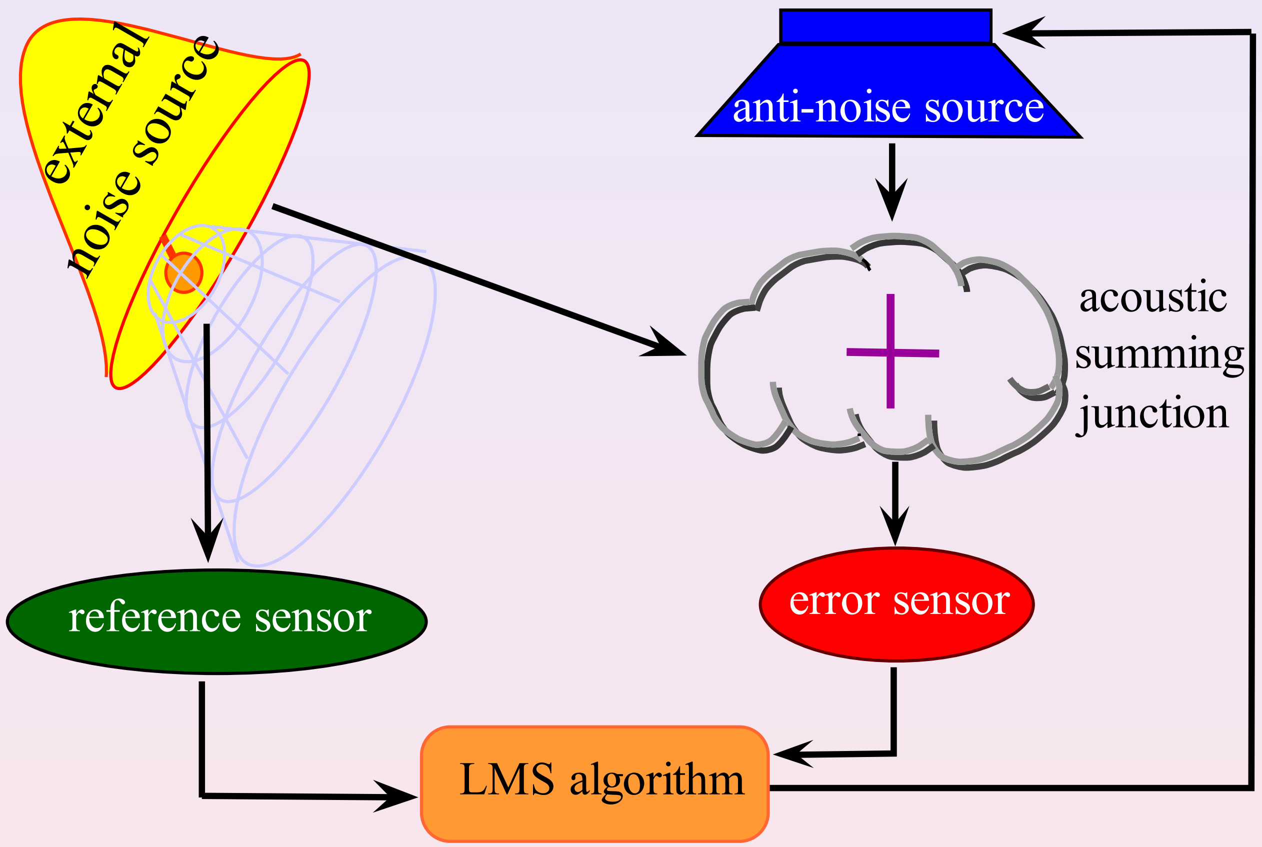
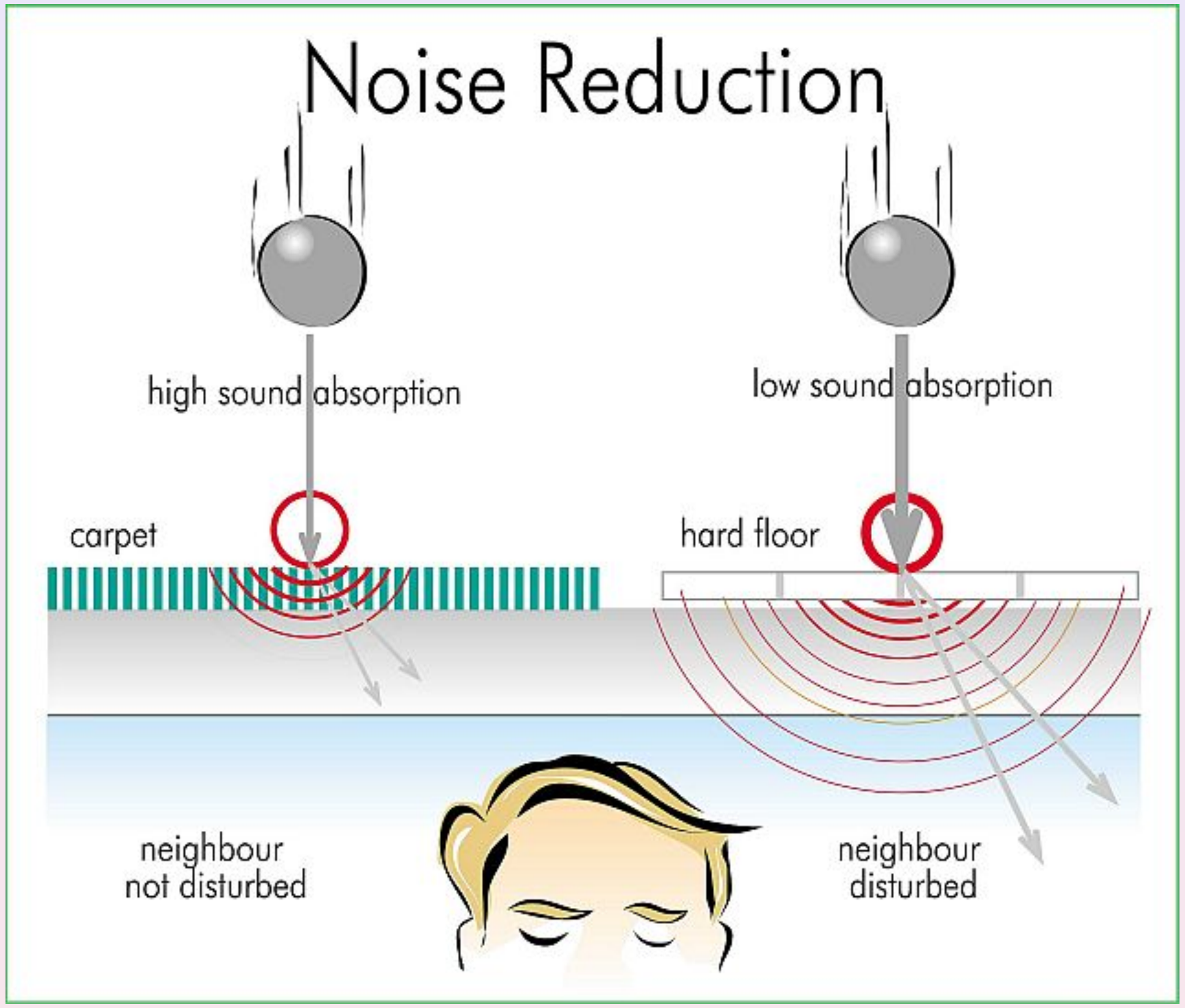
Table of Contents

Motivation: noise reduction





Acoustic noise reduction

Active versus passive controllers.

And many others...

- Noise reduction in cavities and vehicles.
- Laser control in Quantum mechanical and molecular systems.
- Seismic waves, earthquakes.
- Flexible structures.
- Environment: the Thames barrier.
- Optimal shape design in aeronautics.
- Human cardiovascular system: the bypass
- Oil prospection and recovery.
- Irrigation systems.
- ....

The mathematical theory needed to understand these issues combines:

- Partial Differential Equations (PDE)
- Networks and graph theory
- Control Theory
- Optimal Design
- Optimization
- Spectral analysis
- Microlocal analysis
- Numerical analysis
- ...

In this talk we aim to present some toy models and problems, together with some key results and research perspectives.

Table of Contents

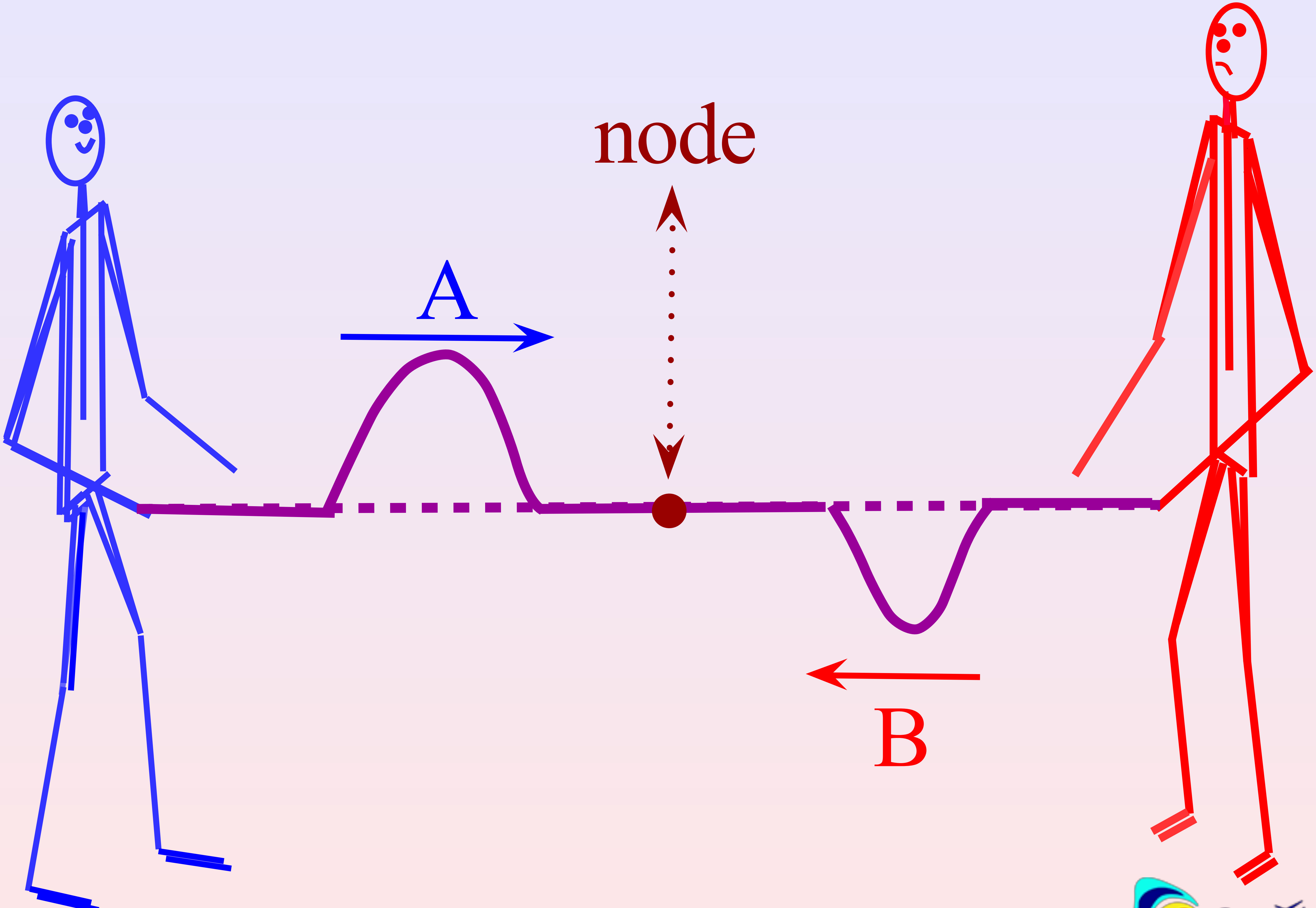
Control of $1-d$ vibrations of a string

The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control on one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

$y = y(x,t)$ is the state and $v = v(t)$ is the control.
The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time T: Given initial data $\{y^0(x), y^1(x)\}$ to find a control $v = v(t)$ such that

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1.$$


node
A
B

The dual observation problem

The control problem above is equivalent to the following one, on the adjoint wave equation:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,0) = \varphi^0(x),\ \varphi_t(x,0) = \varphi^1(x), & 0 < x < 1. \end{cases}$$

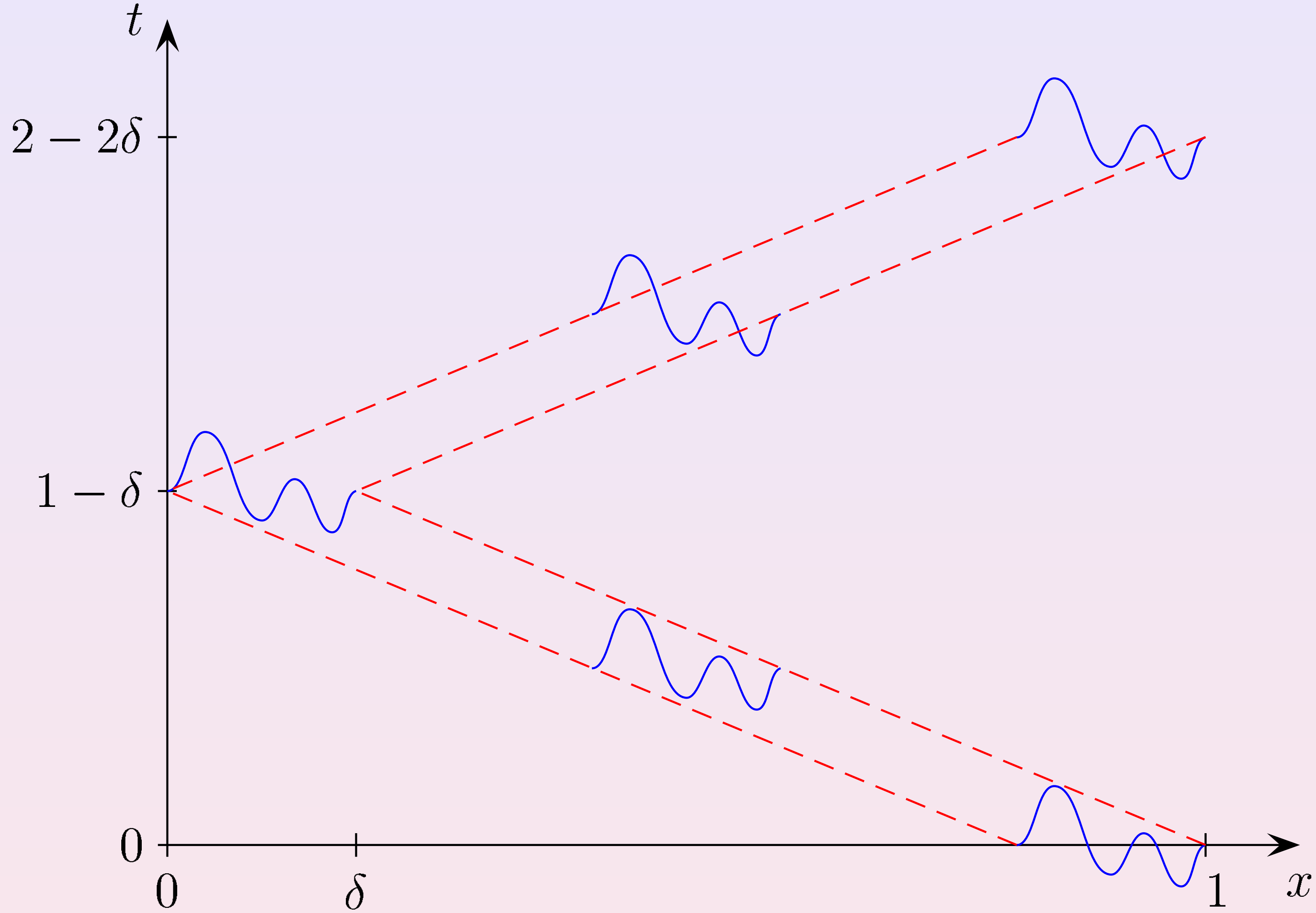
The energy of solutions is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|\varphi_x(x,t)|^2 + |\varphi_t(x,t)|^2\right] dx = E(0), \ \ \forall 0 \le t \le T.$$

The question is then reduced to analyze whether the folllowing inequality is true. This is the so called observability inequality:

$$E(0) \le C(T) \int_0^T |\varphi_x(1,t)|^2\, dt.$$

The answer to this question is easy to gues: The observability inequality holds if and only if $T \geq 2$.



$$E(0) \leq \int_0^T |\varphi_x(1,t)|^2 dt$$

Wave localized at $t=0$ near the extreme $x=1$ propagating with velocity one to the left, bounces on the boundary point $x=0$ and reaches the point of observation $x=1$ in a time of the order of 2.

Construction of the Control

Following J.L. Lions' HUM (Hilbert Uniqueness Method), the control is

$$v(t) = \varphi_x(1, t),$$

where φ is the solution of the adjoint system corresponding to initial data $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1} \times H_0^1},$$

in the space $H_0^1(0,1) \times L^2(0,1)$.
Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that $\varphi_x(1,t) \in L^2(0,T)$ (hidden regularity). Moreover,

COERCIVITY OF J = OBSERVABILITY INEQUALITY. [1]

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[1] Norbert Wiener (1894–1964) defined Cybernetics as the science of control and communication in animals and machines

Outline

Set $h = 1/(N+1) > 0$ and consider the mesh

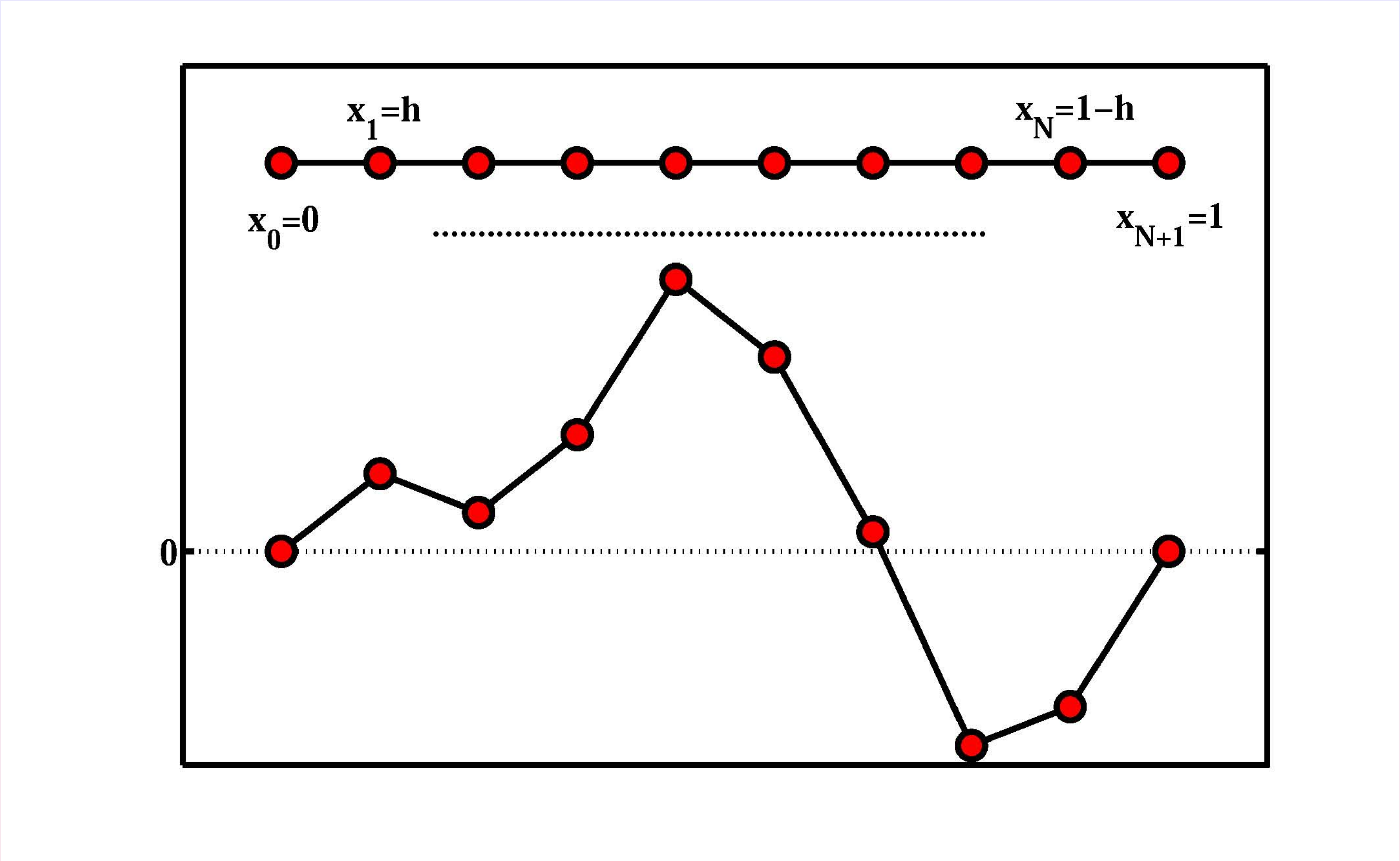
$$x_0 = 0 < x_1 < ... < x_j = jh < x_N = 1 - h < x_{N+1} = 1,$$

which divides $[0, 1]$ into $N + 1$ subintervals

$$I_j = [x_j, x_{j+1}], j = 0, ..., N.$$

Finite difference semi-discrete approximation of the wave equation:

$$\begin{cases} \varphi_j'' - \frac{1}{h^2}\left[\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j\right] = 0, & 0 < t < T,\ j = 1, \dots, N \\ \varphi_j(t) = 0, & j = 0,\ N+1,\ 0 < t < T \\ \varphi_j(0) = \varphi_j^0,\ \varphi_j'(0) = \varphi_j^1, & j = 1, \dots, N. \end{cases}$$



From finite-dimensional dynamical systems to infinite-dimensional ones in purely conservative dynamics.....

Then it should be sufficient to minimize the discrete functional

$$J_h(\varphi^0, \varphi^1) = \frac{1}{2}\int_0^T \frac{|\varphi_N(1,t)|^2}{h^2} dt + h\sum_{j=1}^N \varphi_j^1 y_j^0 - h\sum_{j=1}^N \varphi_j^0 y_j^1,$$

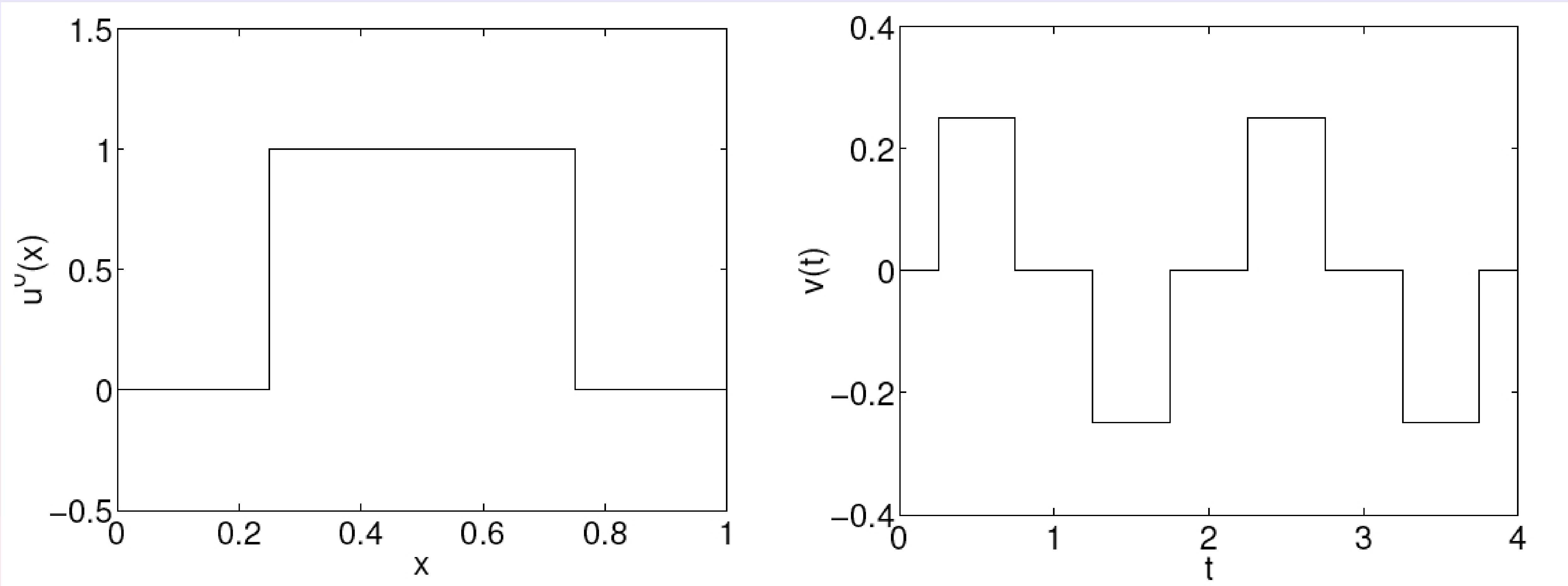
which is a discrete version of the functional J of the continuous wave equation since

$$-\frac{\varphi_N(t)}{h} = \frac{\varphi_{N+1} - \varphi_N(t)}{h} \sim \varphi_x(1,t).$$

Then

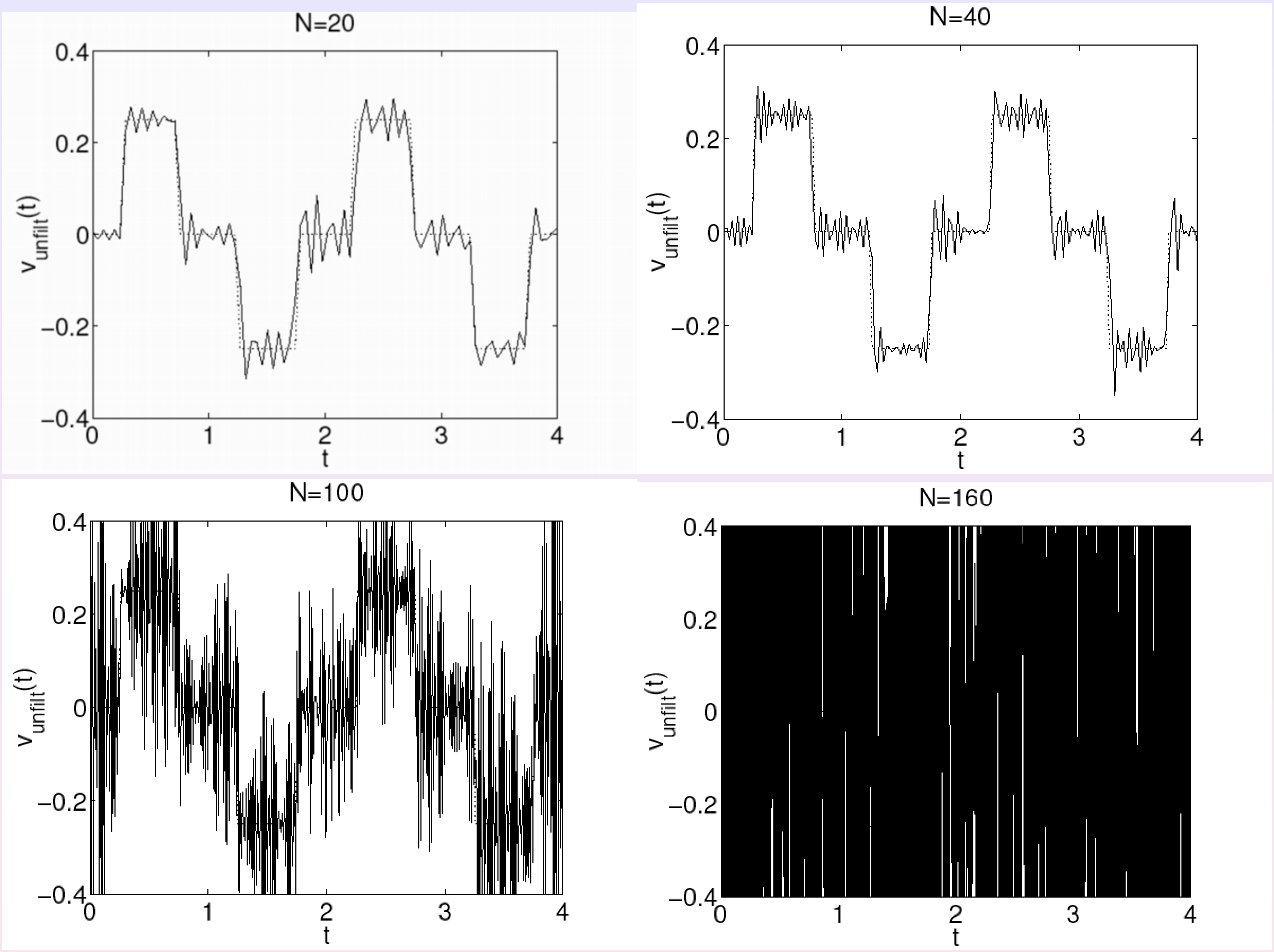
$$v_h(t) = -\frac{\varphi_N^\star(t)}{h}.$$

A NUMERICAL EXPERIMENT



Plot of the initial datum to be controlled for the string occupying the space interval $0 < x < 1$.

Plot of the time evolution of the exact control for the wave equation in time $T = 4$.



The control diverges as $h \to 0$. [2]

[2] E. Z. Propagation, observation, and control of waves approximated by finite difference methods. SIAM Review, 47 (2) (2005), 197-243.

WHY?

The Fourier series expansion shows the analogy between continuous and discrete dynamics.
Discrete solution:

$$\vec{\varphi} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h} t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h} t\right) \right) \vec{w}_k^h.$$

Continuous solution:

$$\varphi = \sum_{k=1}^{\infty} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x)$$

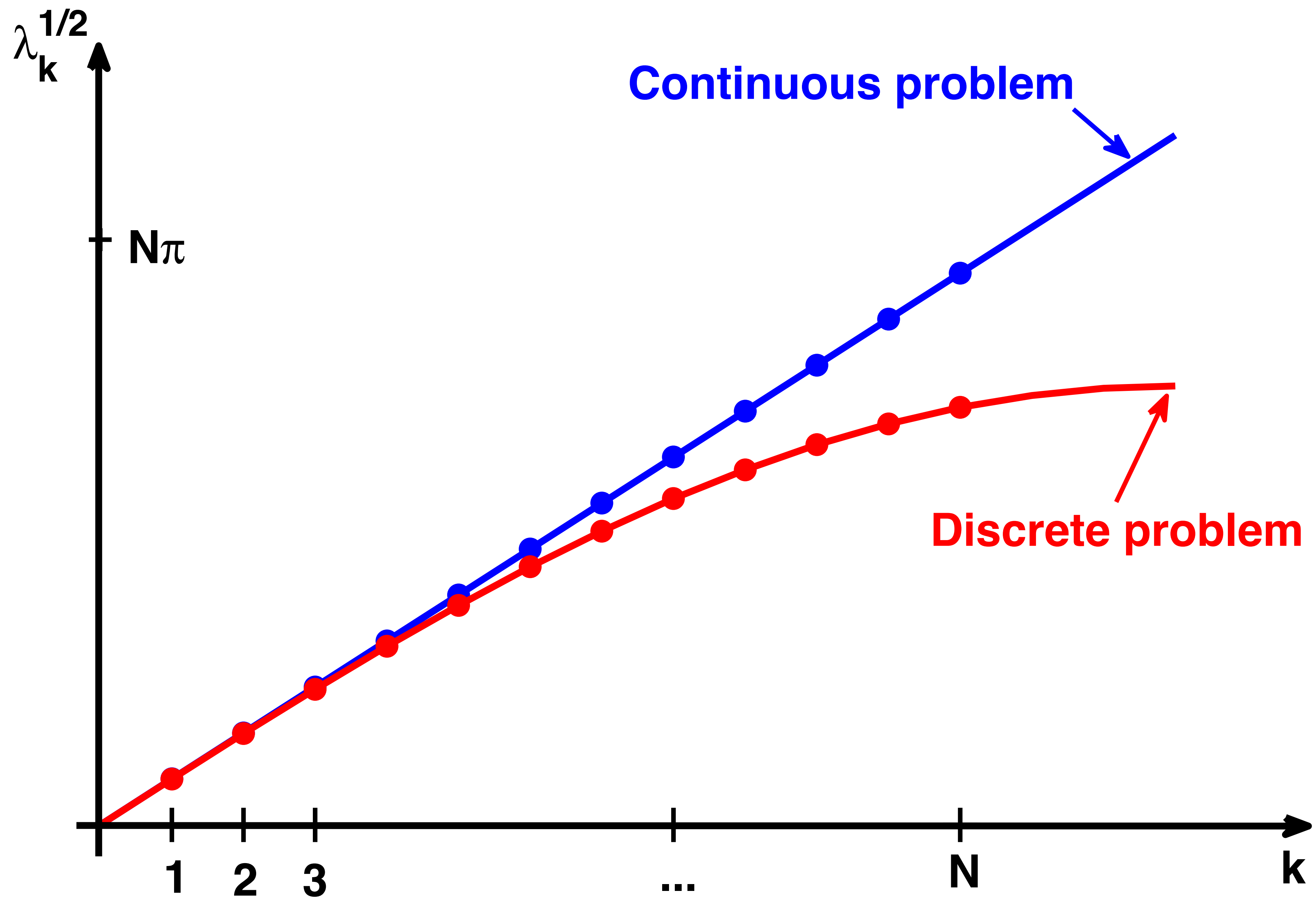
Recall that the discrete spectrum is as follows and converges to the continuous one:

$$\lambda_k^h = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right)$$

$$\lambda_k^h \to \lambda_k = k^2\pi^2, \text{ as } h \to 0$$

$$w_k^h = (w_{k,1}, \ldots, w_{k,N})^T : w_{k,j} = \sin(k\pi jh), \; k, j = 1, \ldots, N.$$

The only relevant differences arise at the level of the dispersion properties and the group velocity. High frequency waves do not propagate, remain captured within the grid, without never reaching the boundary. This makes it impossible the uniform boundary control and observation of the discrete schemes as $h \to 0$.

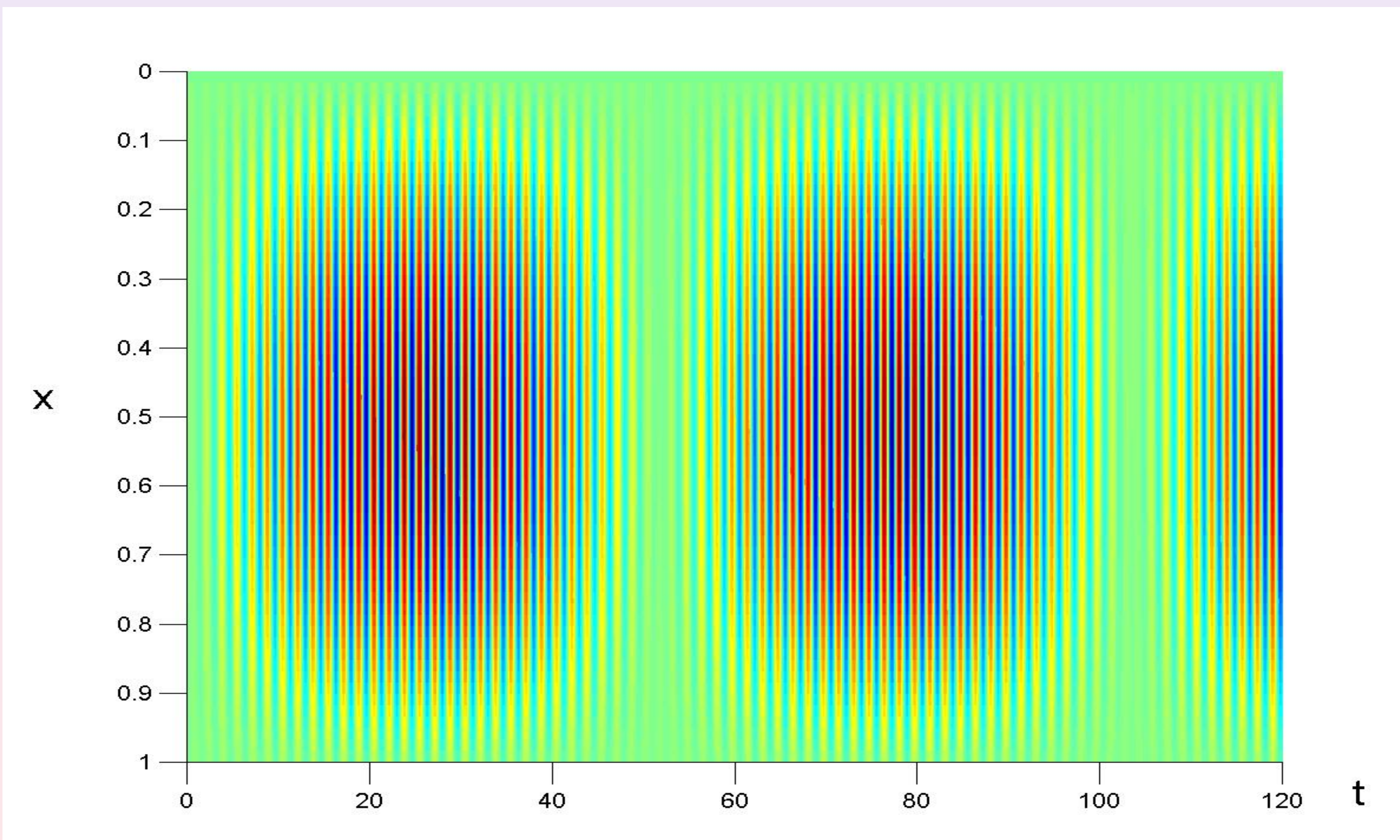


Graph of the square roots of the eigenvalues both in the continuous and in the discrete case. The gap is clearly independent of k in the continuous case while it is of the order of h for large k in the discrete one.

A numerical phamtom

$$\vec{\varphi} = \exp\left(i\sqrt{\lambda_N(h)}\, t\right)\vec{w}_N - \exp\left(i\sqrt{\lambda_{N-1}(h)}\, t\right)\vec{w}_{N-1}.$$

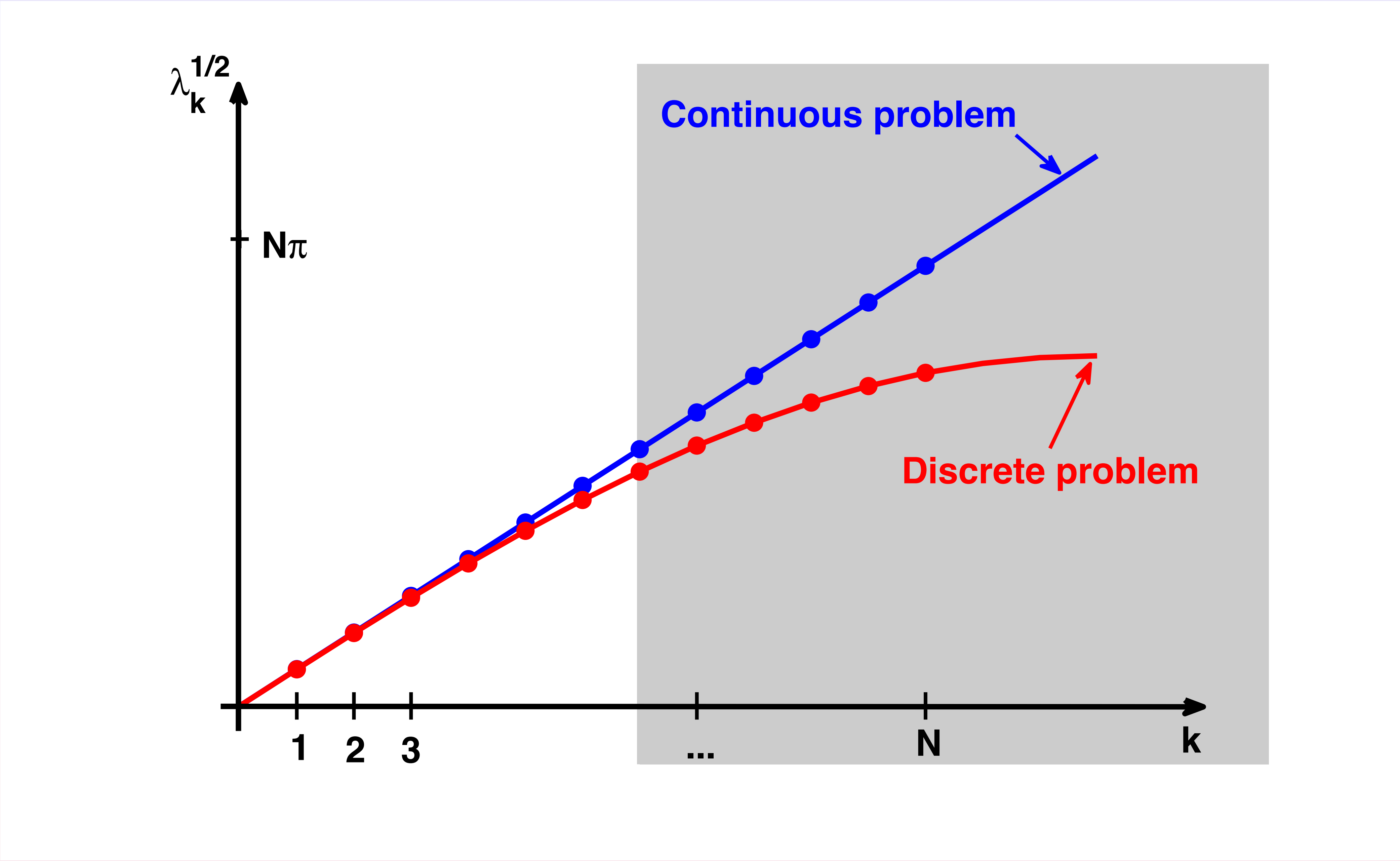
Spurious semi-discrete wave combining the last two eigenfrequencies with very little gap: $\sqrt{\lambda_N(h)} - \sqrt{\lambda_{N-1}(h)} \sim h$.



$h = 1/61$, $(N = 60)$, $0 \leq t \leq 120$.

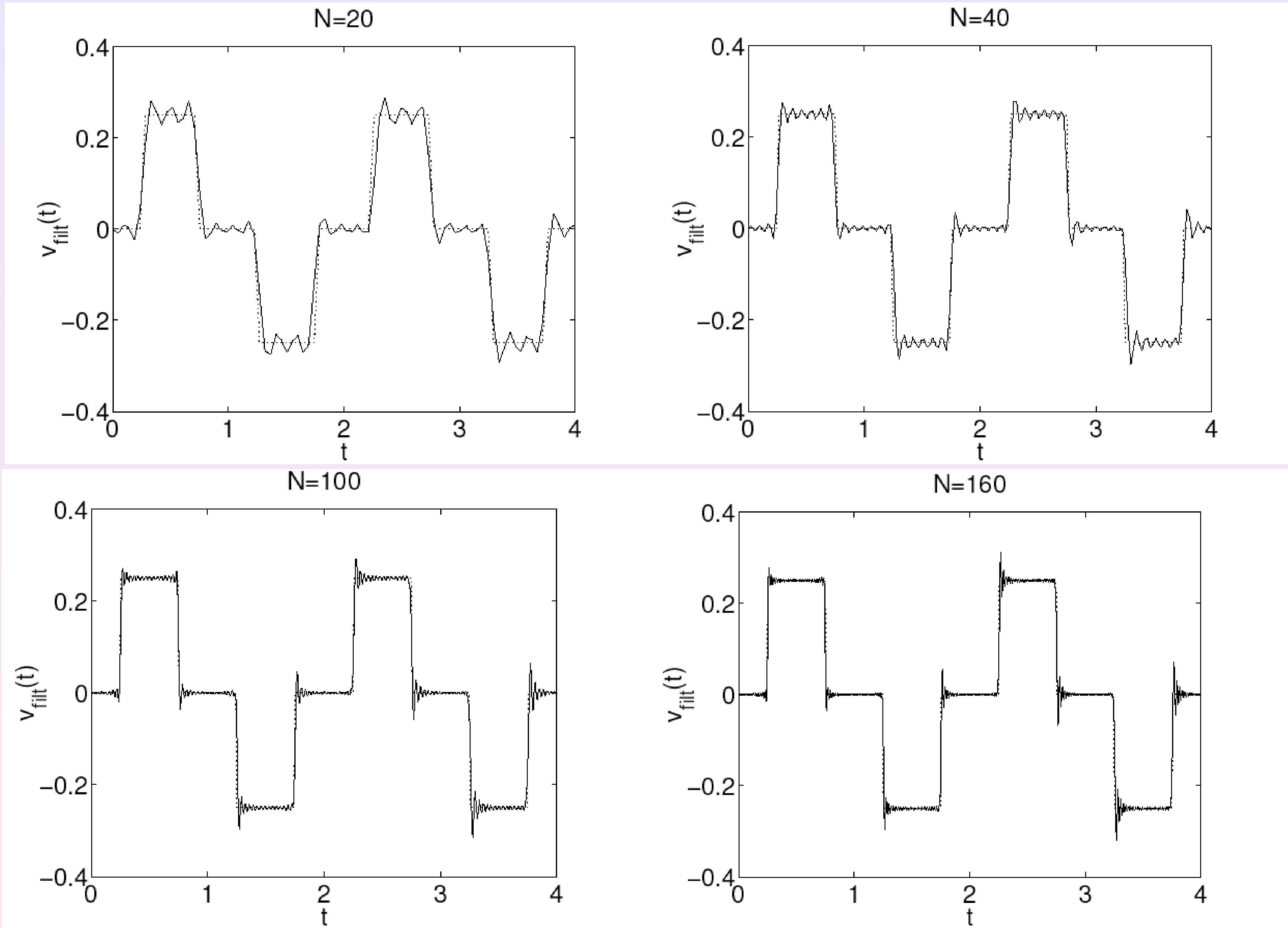
Outline

Fourier filtering



To filter the high frequencies, keeping the components $k \leq \delta/h$ with $0 < \delta < 1$. Then the group velocity remains uniformly bounded below and uniform observation holds in time $T(\delta) > 2$ such that $T(\delta) \to 2$ as $\delta \to 0$.

Numerical experiment, revisited, with filtering



With appropriate filtering the control converges as $h \to 0$.

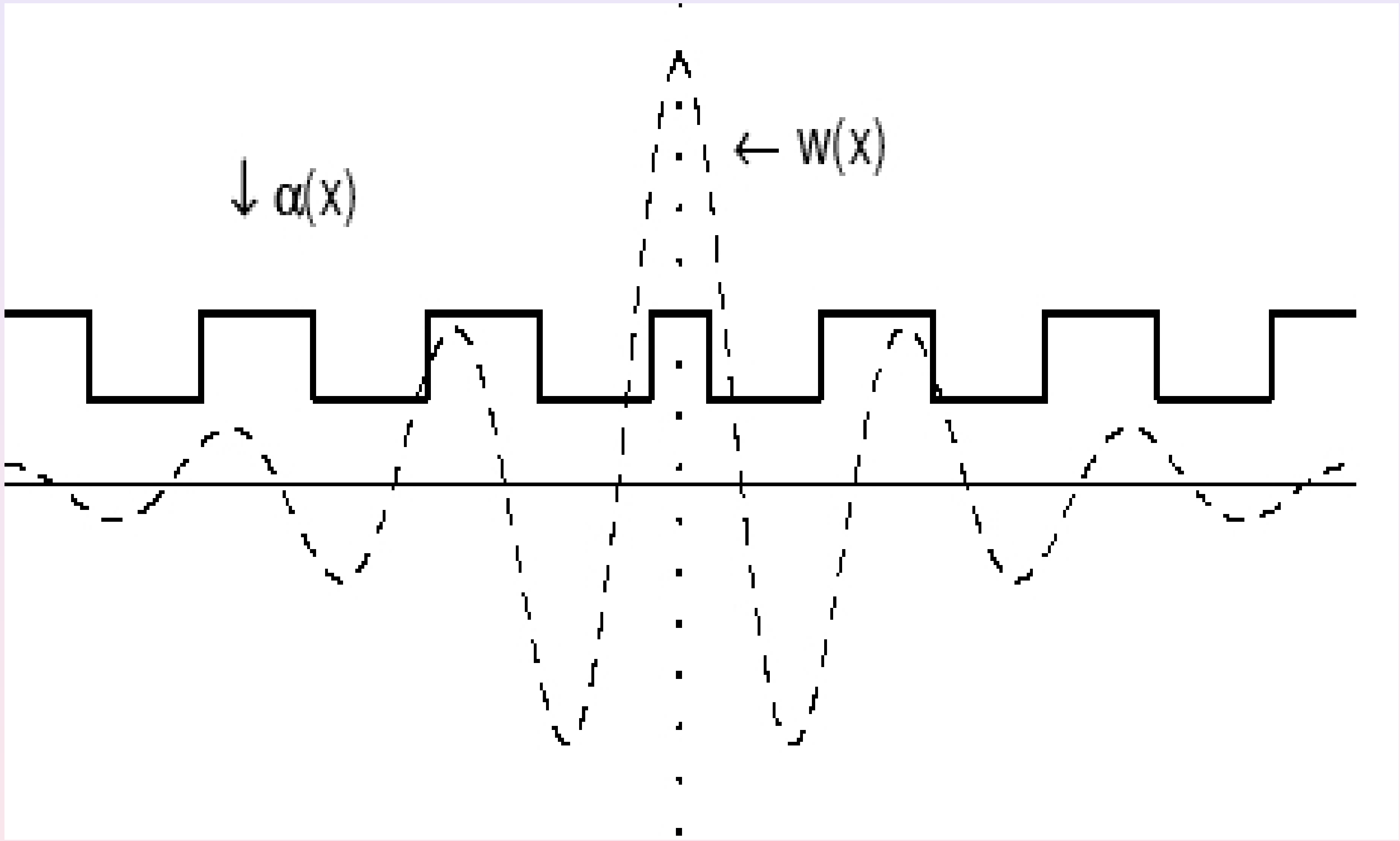
CONCLUSION

- The minima of J_h diverge because its coercivity is vanishing as $h \to 0$;
- This is intimately related to the blow-up of the discrete observability constant $C_h(T) \to \infty$, for all $T > 0$ as $h \to 0$:

$$E_h(0) \leq C_h(T) \int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt$$

- This is due to the lack of propagation of high frequency numerical waves due to the dispersion that the numerical grid produces.
- Actually it is known that $C_h(T)$ diverges exponentially: S. Micu, Numerische Math., 2002.

WELL KNOWN PHENOMENA FOR WAVES IN HIGHLY OSCILLATORY MEDIA



$$\varphi_{tt} - (\alpha(x)\varphi_x)_x = 0.$$

- F. Colombini & S. Spagnolo, Ann. Sci. ENS, 1989
- M. Avellaneda, C. Bardos & J. Rauch, Asymptotic Analysis, 1992.
- C. Castro & E. Z. Archive Rational Mechanics and Analysis, 2002.

DISCRETE MULTIPLIERS

The proof of uniform observability of discrete filtered solutions can developed in various ways:

- Using Ingham inequality in their Fourier series representation since filtering guarantess an uniform gap condition;
- Discrete multipliers:

The multiplier $x\varphi_x$ for the wave equation yields:

$$TE(0) + \int_0^1 x\varphi_x\varphi_t \, dx\Big|_0^T = \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 \, dt.$$

and this implies, as needed,

$$(T-2)E(0) \leq \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 \, dt.$$

The multiplier $j(\varphi_{j+1} - \varphi_{j-1})$ for the discrete wave equation gives:

$$TE_h(0) + X_h(t)\big|_0^T = \frac{1}{2}\int_0^T \left|\frac{\varphi_N(t)}{h}\right|^2 dt + \frac{h}{2}\sum_{j=0}^{N}\int_0^T \mid \varphi_j' - \varphi_{j+1}' \mid^2 dt,$$

Note that

$$\frac{h}{2}\sum_{j=0}^{N}\int_0^T \mid \varphi_j' - \varphi_{j+1}' \mid^2 dt \sim \frac{h^2}{2}\int_0^T\int_0^1 |\varphi_{xt}|^2 dxdt.$$

Filtering is needed to absorb this higher order term: For $1 \leq j \leq \delta N$

$$\left|\frac{h}{2}\sum_{j=0}^{N}\int_0^T \mid \varphi_j' - \varphi_{j+1}' \mid^2 dt\right| \leq \gamma(\delta)\, TE(0),$$

with $0 < \gamma(\delta) < 1$.

Table of Contents

TWO-GRID ALGORITHM (R. Glowinski, M. Asch-G. Lebeau, M. Negreanu, L. Ignat, E. Z.)

To develop on the physical space a different remedy to Fourier filtering. High frequencies producing lack of gap and spurious numerical solutions correspond to large eigenvalues

$$\sqrt{\lambda_N^h} \sim 2/h.$$

When refining the mesh

$$h \to h/2, \qquad \sqrt{\lambda_{2N}^{h/2}} \sim 4/h.$$

Refining the mesh $h \to h/2$ produces the same effect as filtering with parameter $1/2$.

Solutions on the fine grid of size h corresponding to slowly oscillating data given in the coarse mesh ($2h$) are no longer pathological:

$$\varphi = \varphi_l + \varphi_h, \varphi_l = \sum_{k=1}^{(N-1)/2} c_k \vec{w}_k, \varphi_h = \sum_{k=1}^{(N-1)/2} c_k \frac{\lambda_k}{\lambda_{N+1-k}} \vec{w}_{N+1-k},$$

$$||\varphi_h|| \leq ||\varphi_l||.$$

$1 - d$

- M. Negreanu & E. Z., 2004. The two-grid algorithm converges for control times $T > 4$. Multipliers techniques.
- M. Mehrenberger & P. Loreti, 2005. Same result for $T > 2\sqrt{2}$ using Ingham inequalities.

SUMMARY:

- The most natural numerical methods for computing the controls diverge.
- Filtering of the high frequencies is needed. This may be done on the Fourier series expansion or on the physical space by a two-grid algorithm.
- Convergence of the controls is guaranteed by minimizing the discrete functional J_h over the class of slowly oscillating data. This produces a relaxation of the control requirement: only the projection of the discrete state over the coarse mesh vanishes.

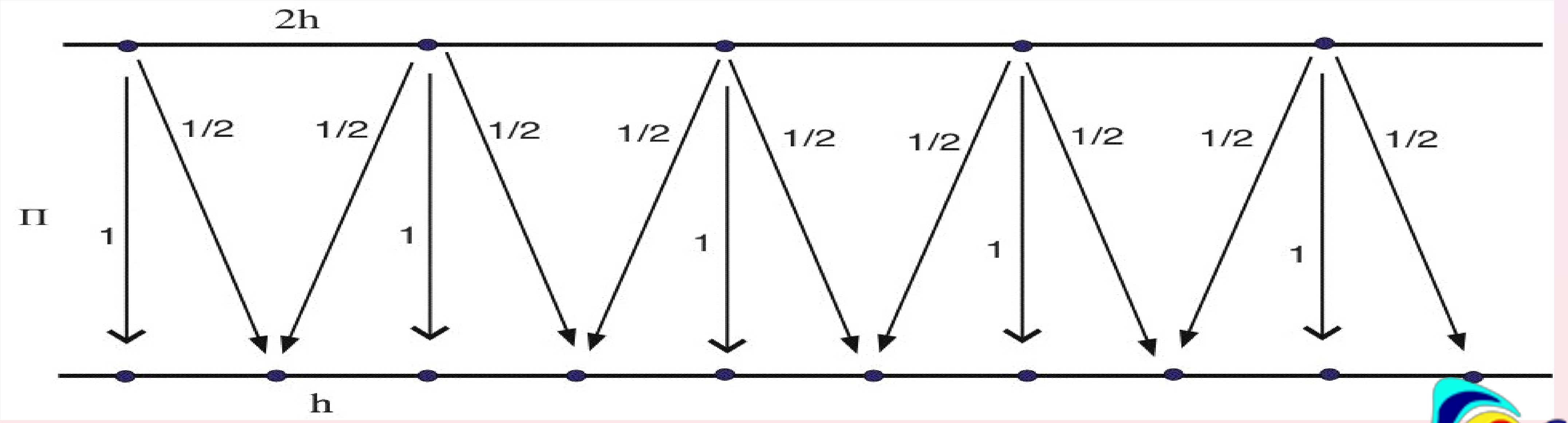


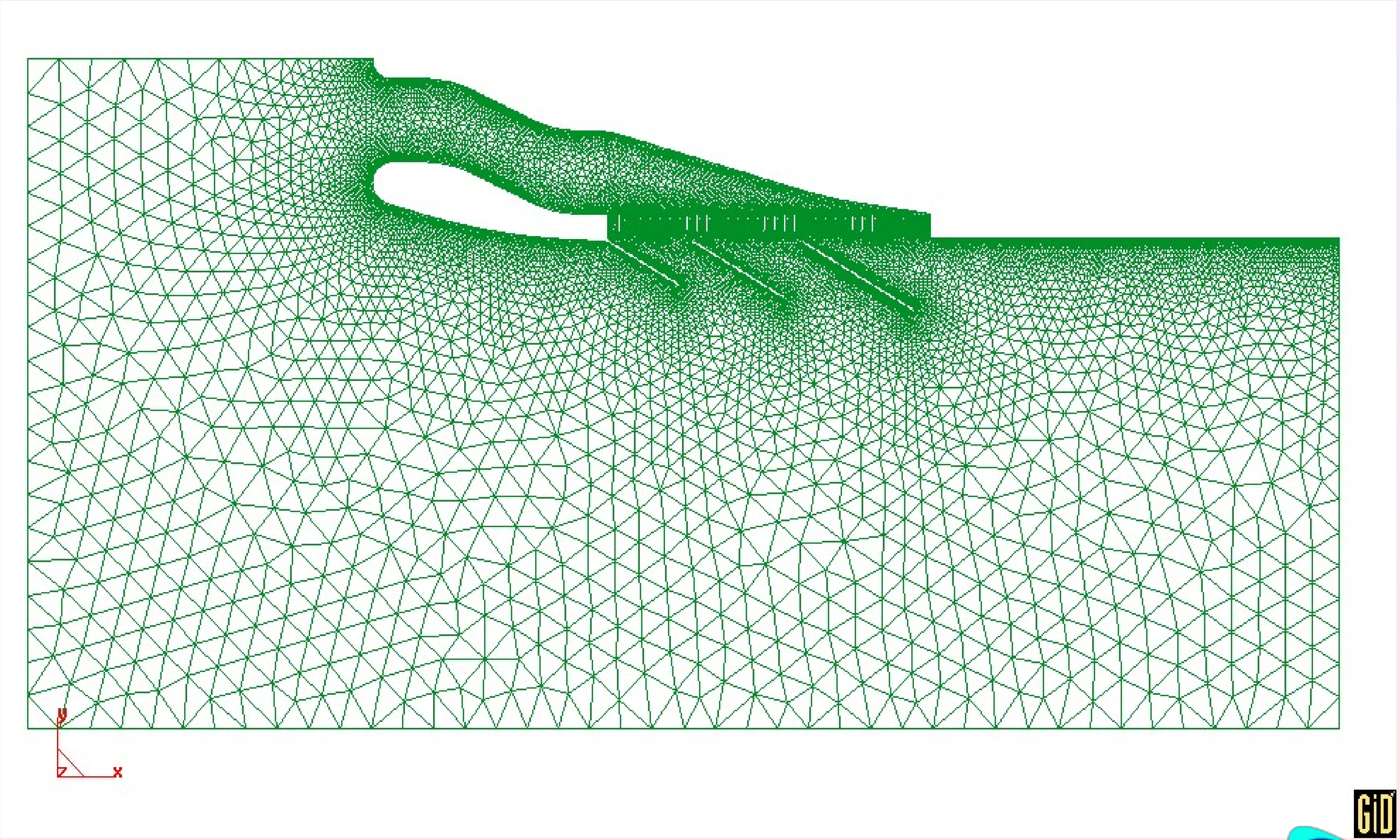
Table of Contents

CONCLUSIONS:

- CONTROL AND NUMERICS DO NOT COMMUTE
- FOURIER FILTERING, MULTI-GRID METHODS ARE GOOD REMEDIES IN SIMPLE SITUATIONS: CONSTANT COEFFICIENTS, REGULAR MESHES.
- MUCH REMAINS TO BE DONE TO HAVE A COMPLETE THEORY AND TO HANDLE MORE COMPLEX SYSTEMS. BUT ALL THE PATHOLOGIES WE HAVE DESCRIBED WILL NECESSARILY ARISE IN THOSE SITUATIONS TOO.
- THE MATHEMATICAL THEORY NEEDS TO COMBINE TOOLS FROM PARTIAL DIFFERENTIAL EQUATIONS, CONTROL THEORY, CLASSICAL NUMERICAL ANALYSIS AND MICROLOCAL ANALYSIS.

OPEN PROBLEMS

Complex geometries, variable and irregular coefficients, irregular meshes, the system of elasticity, nonlinear state equations, ...



To learn more on this topic:

- E. Z. Propagation, observation, and control of waves approximated by finite difference methods. SIAM Review, 47 (2) (2005), 197-243.
- S. ERVEDOZA and E. Z., The Wave Equation: Control and Numerics, in "Control and stabilization of PDE's", P. M. Cannarsa and J. M. Coron, eds., 'Lecture Notes in Mathematics", CIME Subseries, Springer Verlag, to appear.
- A. MARICA and E. Z., Symmetric discontinuous Galerkin approximations of $1-d$ waves: High frequency propagation and observability, Springer Briefs, to appear.
- L. IGNAT & E. Z., Dispersive Properties of Numerical Schemes for Nonlinear Schrödinger Equations, Proceedings of FoCM'2005, Santander, June-July 2005.