Well-posed systems - a fast review of the basics

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The concept of well-posed system

Informally speaking, a system is *well-posed* if on any time interval $[\tau, t]$, for any initial state x_0 in the state space and any input function u in a specified space of functions, it has a unique state trajectory x and a unique output function y, both defined on $[\tau, t]$. Moreover, y must belong to a specified space of functions, and both $x(t)$ and y must depend continuously on $x(\tau)$ and on u. This concept is general and can be made precise for many classes of non-linear and/or time-varying systems. However, most attention in the literature has been devoted to the simplest particular case, namely, linear and time-invariant (LTI) systems, because here we have strong tools to develop the theory.

Well-posed systems - the LTI context

In the LTI context, if the state space is finite-dimensional, then well-posedness is not an issue and is usually not even mentioned. There is a meaningful LTV theory, less developed. The theory focuses on systems with an infinite-dimensional state space, usually a Hilbert space. This is motivated by a variety of systems described by partial differential or delay equations, that can be shown to fit into this framework. The idea of the definition is that the system is fully described by the operators Σ_τ that map $\begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}$ to $\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix}$, where x is the state trajectory of the system, u, y are the input and output signals, and $\mathbf{P}_\tau$ is projection by truncation to the interval $[0, \tau]$. We partition these operators as follows:

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix} \qquad \forall\, \tau \geq 0. \tag{1}$$

The definition of well-posed LTI systems

The definition will list the requirements that have to be imposed on each of the four component operator families, so that the concept corresponds to what we expect based on our intuition and experience.

Notation. Let W be a Hilbert space. For any interval J, we regard $L^2_{\rm loc}(J;W)$ as a subspace of $L^2_{\rm loc}(\mathbb{R};W)$ (by extending functions defined on J with the value 0 outside J). The operator of truncation onto J is denoted $\mathbf{P}_J$ and the bilateral right shift operator by $\tau \in \mathbb{R}$ is denoted $\mathcal{S}_\tau$. For any $u, v \in L^2_{\rm loc}([0,\infty);W)$ and any $\tau \geq 0$, the τ*-concatenation* of u and v is the function defined by

$$u \underset{\tau}{\diamondsuit} v = \mathbf{P}_{[0,\tau]} u + \mathcal{S}_\tau v \, .$$

Thus, $(u \underset{\tau}{\diamondsuit} v)(t) = u(t)$ for $t \in [0,\tau)$, while $(u \underset{\tau}{\diamondsuit} v)(t) = v(t-\tau)$ for $t \geq \tau$. If $\mathbb{T}$ is an *operator semigroup* (C_0-semigroup), we denote its growth bound by $\omega_0(\mathbb{T})$.

The formal definition

Let U, X and Y be Hilbert spaces. A *well-posed linear system* is a family of operators $\Sigma = (\Sigma_t)_{t\geq 0}$ partitioned as in (1), where

1. $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ is an operator semigroup on X,
2. $\Phi = (\Phi_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2([0,\infty); U)$ to X such that

$$\Phi_{\tau+t}(u \underset{\tau}{\diamondsuit} v) = \mathbb{T}_t \Phi_\tau u + \Phi_t v, \tag{2}$$

   for every $u, v \in L^2([0,\infty); U)$ and all $\tau, t \geq 0$,
3. $\Psi = (\Psi_t)_{t\geq 0}$ is a family of bounded linear operators from X to $L^2([0,\infty); Y)$ such that

$$\Psi_{\tau+t} x_0 = \Psi_\tau x_0 \underset{\tau}{\diamondsuit} \Psi_t \mathbb{T}_\tau x_0, \tag{3}$$

   for every $x_0 \in X$ and all $\tau, t \geq 0$, and $\Psi_0 = 0$,

The formal definition - continued

4. $\mathbb{F} = (\mathbb{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2([0,\infty); U)$ to $L^2([0,\infty); Y)$ such that

$$\mathbb{F}_{\tau+t}(u \underset{\tau}{\diamondsuit} v) = \mathbb{F}_\tau u \underset{\tau}{\diamondsuit} (\Psi_t \Phi_\tau u + \mathbb{F}_t v), \tag{4}$$

for every $u, v \in L^2([0,\infty); U)$ and all $\tau, t \geq 0$, and $\mathbb{F}_0 = 0$.

We call U the *input space*, X the *state space* and Y the *output space* of Σ. The operators Φ_τ are called *input maps*, the operators Ψ_τ are called *output maps*, and the operators $\mathbb{F}_\tau$ are called *input-output maps*.

There are several equivalent definitions, e.g., Olof Staffans uses a different one in his book, that is based on operators acting on infinite time intervals. There is an interesting equivalent definition that reduces well-posed systems to a special class of operator semigroups, called Lax-Phillips semigroups. We do not explain this here.

The delay line

As an exercise, the reader may construct the well-posed linear system corresponding to a set of matrices A, B, C, D of compatible dimensions, so that the classical equations $\dot{x}(t) = Ax(t) + Bu(t),\ \ y(t) = Cx(t) + Du(t)$ make sense.

We give an extremely simple but important example of an infinite-dimensional well-posed system. We model a delay line as a well-posed linear system. Let $X = L^2[-h, 0]$, where $h > 0$, and let $\mathbb{T}$ be the left shift semigroup on X with zero entering from the right, i.e., for any $\tau \geq 0$ and $\zeta \in [-h, 0]$,

$$(\mathbb{T}_\tau x)(\zeta) = \begin{cases} x(\zeta + \tau), & \text{for} \quad \zeta + \tau \leq 0, \\ 0, & \text{for} \quad \zeta + \tau > 0. \end{cases}$$

Let $U = \mathbb{C}$ and for any $\tau \geq 0$ and $\zeta \in [-h, 0]$ define

$$(\Phi_\tau u)(\zeta) = \begin{cases} u(\zeta + \tau), & \text{for} \quad \zeta + \tau \geq 0, \\ 0, & \text{for} \quad \zeta + \tau < 0. \end{cases}$$

The delay line - continued

Let $Y = \mathbb{C}$ and for any $\tau \geq 0$ and $t \in [0, \tau)$ define

$$(\Psi_\tau x_0)(t) = \begin{cases} x(t-h), & \text{for} \quad t-h \leq 0, \\ 0, & \text{for} \quad t-h > 0. \end{cases}$$

For $t \geq \tau$ we put $(\Psi_\tau x)(t) = 0$. Finally, let for any $\tau \geq 0$ and $t \in [0, \tau)$

$$(\mathbb{F}_\tau u)(t) = \begin{cases} u(t-h), & \text{for} \quad t-h \geq 0, \\ 0, & \text{for} \quad t-h < 0. \end{cases}$$

For $t \geq \tau$ we put $(\mathbb{F}_\tau u)(t) = 0$. Then $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system. It is clear from the formula of $\mathbb{F}$ that this is indeed a delay line of size h. Its transfer function (to be defined soon) is $\mathbf{G}(s) = e^{-hs}$.

The spaces X_1 and X_{-1}

Let $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ be a strongly continuous semigroup of bounded linear operators on the Hilbert space X, with generator A. We define on X a new norm by

$$\|x\|_{-1} = \|(\beta I - A)^{-1} x\|,$$

where $\beta \in \rho(A)$ is fixed. The choice of β is not important, because different choices lead to equivalent norms. The generator A determines two additional Hilbert spaces as follows: X_1 is $\mathcal{D}(A)$ with the norm $\|x\|_1 = \|(\beta I - A)x\|$ (this norm is equivalent to the graph norm), while X_{-1} is the completion of X with respect to the norm $\| \cdot \|_{-1}$. It is possible to extend A to an operator $A_{-1} \in \mathcal{L}(X, X_{-1})$ which generates a strongly continuous semigroup $\mathbb{T}_{-1}$ on X_{-1}. Thus,

$$X_1 \subset X \subset X_{-1}.$$

The representation of the operators Φ_t

The spaces X_1^d and X_{-1}^d are defined similarly, but with A^* in place of A. Then X_{-1} is the dual of X_1^d, with pivot X.

A nontrivial consequence of assumptions (i) and (ii) in the definition is that there exists a unique $B \in \mathcal{L}(U, X_{-1})$, called the *control operator* of Σ, such that

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) \mathrm{d}\sigma \qquad \forall\, t \geq 0. \tag{5}$$

Notice that in the above formula, $\mathbb{T}$ acts on X_{-1} and the integration is carried out in X_{-1}. $\Phi_t u$ depends continuously on t. The operator B can be found by

$$B\mathrm{v} = \lim_{\tau \to 0} \frac{1}{\tau} \Phi_\tau(\chi \cdot \mathrm{v}) \qquad \forall\, \mathrm{v} \in U, \tag{6}$$

where χ denotes the characteristic function of the interval $[0, \infty)$. B is an *admissible control operator* for $\mathbb{T}$.

The representation of the operators Ψ_t

Now we turn our attention to the output maps of the well-posed system, the operators that map the initial state to segments of the output function. It follows from the identity $\mathbf{P}_{[0,\tau]}\Psi_{\tau+t} = \Psi_\tau$ (for $\tau, t \geq 0$) that there exists a unique operator $\Psi_\infty : X \to L^2_{\rm loc}([0,\infty); Y)$ such that $\mathbf{P}_{[0,\tau]}\Psi_\infty = \Psi_\tau$ for all $\tau \geq 0$. Ψ_∞ is called the *extended output map* of Σ, and it satisfies

$$\Psi_\infty x_0 = \Psi_\infty x_0 \underset{\tau}{\diamondsuit} \Psi_\infty \mathbb{T}_\tau x_0 , \tag{7}$$

for every $x_0 \in X$ and all $\tau \geq 0$. It can be shown (using assumptions 1 and 3 in the definition) that there exists a unique $C \in \mathcal{L}(X_1, Y)$, called the *observation operator* of Σ, such that for every $x_0 \in \mathcal{D}(A)$ and all $t \geq 0$,

$$(\Psi_\infty x_0)(t) = C\mathbb{T}_t x_0 . \tag{8}$$

This determines Ψ_∞, since $\mathcal{D}(A)$ is dense in X.

The representation of the operators Ψ_t - continued

An operator $C \in \mathcal{L}(X_1, Y)$ is an *admissible observation operator* for $\mathbb{T}$ if the estimate

$$\int_0^\tau \|C\mathbb{T}_t x_0\|^2 \mathrm{d}t \leq k\|x_0\|^2$$

holds for some (hence, for every) $\tau > 0$ and for every $x_0 \in \mathcal{D}(A)$. The constant $k \geq 0$ may depend on τ. If $C \in \mathcal{L}(X, Y)$ then obviously it is admissible. Such observation operators are called *bounded*, while the others are called *unbounded*. It is clear that if C is the observation operator of a well-posed linear system Σ, then C is admissible for the semigroup $\mathbb{T}$ of Σ.

There is a similar terminology for control operators: B is called *bounded* if $B \in \mathcal{L}(U, X)$, and *unbounded* otherwise.

Duality: C is an admissible observation operator for $\mathbb{T}$ if and only if C^* is an admissible control operator for $\mathbb{T}^*$.

The representation of the operators $\mathbb{F}_t$

Now we turn our attention to the input-output maps of the well-posed system Σ. It follows from the identity $\mathbf{P}_{[0,\tau]}\mathbb{F}_{\tau+t} = \mathbb{F}_\tau$ that there exists a unique linear operator $\mathbb{F}_\infty : L^2_{\mathrm{loc}}([0,\infty); U) \to L^2_{\mathrm{loc}}([0,\infty); Y)$ such that $\mathbf{P}_{[0,\tau]}\mathbb{F}_\infty = \mathbb{F}_\tau$ for all $\tau \geq 0$. This $\mathbb{F}_\infty$ is called the *extended input-output map* of Σ. We have

$$\mathbb{F}_\infty(u \underset{\tau}{\diamondsuit} v) = \mathbb{F}_\infty u \underset{\tau}{\diamondsuit} (\Psi_\infty \Phi_\tau u + \mathbb{F}_\infty v), \qquad (9)$$

for every $u, v \in L^2_{\mathrm{loc}}([0,\infty); U)$ and all $\tau \geq 0$. Taking $u = 0$ in (9) we get that

$$\mathbb{F}_\infty \mathcal{S}_\tau = \mathcal{S}_\tau \mathbb{F}_\infty, \qquad (10)$$

for every $\tau \geq 0$. This property means that $\mathbb{F}_\infty$ is *shift-invariant* or *time-invariant*.

The representation of the operators $\mathbb{F}_t$ - continued

Notation. For any Hilbert space W, any interval J and any $\omega \in \mathbb{R}$ we put

$$L^2_\omega(J; W) = e_\omega L^2(J; W),$$

where $(e_\omega v)(t) = e^{\omega t} v(t)$, with the norm $\|e_\omega v\|_{L^2_\omega} = \|v\|_{L^2}$.

We can represent $\mathbb{F}_\infty$ via the *transfer function* **G** of Σ, which is a bounded analytic $\mathcal{L}(U, Y)$-valued function on the half-plane where $\operatorname{Re} s > \omega$, for every $\omega > \omega_0(\mathbb{T})$. If $u \in L^2_\omega([0, \infty); U)$ with $\omega > \omega_0(\mathbb{T})$, then the Laplace integral of $\mathbb{F}_\infty u$ converges absolutely for $\operatorname{Re} s > \omega_0(\mathbb{T})$ and

$$\widehat{(\mathbb{F}_\infty u)}(s) = \mathbf{G}(s)\hat{u}(s), \qquad \operatorname{Re} s > \omega_{\mathbb{F}} . \tag{11}$$

The transfer function **G** satisfies

$$\begin{aligned} \mathbf{G}(s) - \mathbf{G}(\beta) &= (\beta - s)C(\beta I - A)^{-1}(sI - A)^{-1}B \\ &= C\left[(sI - A)^{-1} - (\beta I - A)^{-1}\right]B, \end{aligned} \tag{12}$$

The representation of the operators $\mathbb{F}_t$ - continued

for all s, β with $\mathrm{Re}\, s, \mathrm{Re}\, \beta > \omega_0(\mathbb{T})$. Equivalently,

$$\mathbf{G}'(s) = -C(sI - A)^{-2}B.$$

This shows that **G** is determined by A, B and C up to an additive constant operator. Analytic functions that are bounded on some right half-plane are called *proper*.

Realization theory: Any proper $\mathcal{L}(U, Y)$-valued function has realizations as the transfer function of a well-posed linear system. (There is a theory of minimal realizations, various canonical realizations, scattering passive realizations.)

The operators $\mathbb{F}_t$ can be represented also in time domain, but this is a tricky story - more about this a little later.

Transformations on well-posed systems

There are transformations which lead from one well-posed system to another: duality, time-inversion, flow-inversion and time-flow inversion, the reciprocal transformation. The meaning of time-inversion and flow inversion is clear, and of course they are not always applicable. Time-flow inversion is a combination of time inversion and flow inversion, but it can be applied also for some systems that are neither time nor flow invertible. The reciprocal transformation essentially corresponds to replacing the time t with $1/t$.

Here we briefly recall only the duality transformation.

Notation. Let W be a Hilbert space. For every $u \in L^2_{\text{loc}}([0,\infty); W)$ and all $\tau \geq 0$, we define the *time-inversion operator* on $[0,\tau]$ as follows:

$$(\text{Я}_\tau u)(t) = \begin{cases} u(\tau - t) & \text{for } \; t \in [0,\tau], \\ 0 & \text{for } \; t > \tau. \end{cases}$$

The dual system

Theorem 1. Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ be a well-posed linear system with input space U, state space X and output space Y. Define Σ_τ^d (for all $\tau \geq 0$) by

$$\Sigma_\tau^d = \begin{bmatrix} \mathbb{T}_\tau^d & \Phi_\tau^d \\ \Psi_\tau^d & \mathbb{F}_\tau^d \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}_\tau^* & \Psi_\tau^* \\ \Phi_\tau^* & \mathbb{F}_\tau^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix}. \tag{13}$$

Then $\Sigma^d = (\mathbb{T}^d, \Phi^d, \Psi^d, \mathbb{F}^d)$ is a well-posed linear system with input space Y, state space X and output space U. If A, B and C are the semigroup generator, control operator and observation operator of Σ, then the corresponding operators for Σ^d are A^*, C^* and B^*. The transfer functions are related by

$$\mathbf{G}^d(s) = \mathbf{G}^*(\overline{s}), \qquad \mathrm{Re}\, s > \omega_0(\mathbb{T}).$$

The above system Σ^d is called the *dual system* of Σ.

Motivation for system nodes

Let U, X, Y be Hilbert spaces. We consider an operator

$$S : \mathcal{D}(S) \to X \oplus Y, \quad \text{with } \mathcal{D}(S) \subset X \oplus U,$$

whose intuitive meaning is that it defines a system via the equations

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geq 0. \tag{14}$$

It is not clear when this system of equations has solutions. Based on intuition gained from finite-dimensional linear systems, we expect to associate to a well-posed system Σ an operator of the form

$$S = \begin{bmatrix} A & B \\ C & ? \end{bmatrix},$$

where A, B, C are the operators encountered earlier. We would like the solutions of (14) to be a dense subspace of the trajectories of Σ. This is actually possible, as we shall see.

System nodes

Definition. The operator

$$S : \mathcal{D}(S) \to X \oplus Y\,, \quad \text{with } \mathcal{D}(S) \subset X \oplus U\,,$$

is called a *system node* on (U, X, Y) if it has the following properties:

1. S is closed (as an operator from $X \oplus U$ to $X \oplus Y$).
2. We partition $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. The operator $A : \mathcal{D}(A) \to X$ defined by

   $$Ax = A\&B \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \qquad \mathcal{D}(A) = \left\{x \in X \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{D}(S)\right\}$$

   is the generator of a strongly continuous semigroup on X.
3. The operator $A\&B$ (with $\mathcal{D}(A\&B) = \mathcal{D}(S)$) can be extended to an operator $\begin{bmatrix} A_{-1} & B \end{bmatrix} \in \mathcal{L}(X \oplus U, X_{-1})$.
4. $\mathcal{D}(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X \oplus U \;\middle|\; A_{-1}x + Bu \in X \right\}$.

Note that a priori, the operator $C\&D$ cannot be split.

System nodes - continued

It is easy to see that if S is a system node on (U, X, Y), then $\mathcal{D}(S)$ is dense in $X \oplus U$ and $A\&B$ is closed. Hence, the graph norm on $\mathcal{D}(S)$ is equivalent to the graph norm of the operator $A\&B$ on the same domain, defined by

$$\left\| \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\|^2_{\mathcal{D}(S)} = \|x\|^2 + \|u\|^2 + \|A_{-1}x + Bu\|^2 .$$

The operator A is called the *semigroup generator* of S and B is called the *control operator* of S. The operator $C \in \mathcal{L}(X_1, Y)$ defined by

$$Cx = C\&D \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \qquad \forall\, x \in \mathcal{D}(A)$$

is called the *observation operator* of S. In the sequel we usually write A instead of A_{-1}. The *transfer function* of S is the $\mathcal{L}(U, Y)$-valued analytic function defined by

$$\mathbf{G}(s) = C\&D \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} \qquad \forall\, s \in \rho(A).$$

System nodes - continued

It is easy to see that for all $s, \beta \in \rho(A)$ we have

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C\left[(sI - A)^{-1} - (\beta I - A)^{-1}\right] B, \qquad (15)$$

or equivalently, $\mathbf{G}'(s) = -C(sI - A)^{-2}B$.

Thus, system node is determined by three operators A, B, C acting between the correct spaces (the same as for a well-posed system) and a transfer function $\mathbf{G}$ compatible with A, B, C in the sense of (15), but without any admissibility or well-posedness or properness assumption.

The space Z is defined by

$$Z = X_1 + (\beta I - A)^{-1} BU,$$

where $\beta \in \rho(A)$. The choice of β is not important. Z consists of all possible first components of $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}(S) \subset X \times U$.

Compatible system nodes

The system node is called *compatible* if C has a continuous extension to an operator $\overline{C} \in \mathcal{L}(Z, Y)$. In this case, we may define the operator $D \in \mathcal{L}(U, Y)$ by $D = \mathbf{G}(\beta) - \overline{C}(\beta I - A)^{-1}B$ and it follows from (15) that D is independent of $\beta \in \rho(A)$. Then $C\&D$ and S can be split as in finite dimensions:

$$C\&D \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] = \overline{C}x + Dv\,, \qquad S = \begin{bmatrix} A & B \\ \overline{C} & D \end{bmatrix}$$

and

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B + D \qquad \forall\, s \in \rho(A)\,.$$

A system node S is usually associated with the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geq 0\,, \tag{16}$$

or equivalently,

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \qquad y(t) = C\&D \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geq 0\,.$$

Classical solutions

A triple (x, u, y) is called a *classical solution* of (16) on $[0, \infty)$ if

(a) $x \in C^1([0, \infty); X)$,

(b) $u \in C([0, \infty); U)$, $\; y \in C([0, \infty); Y)$,

(c) $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S)$ for all $t \geq 0$,

(d) (16) holds for all $t \geq 0$.

We remark that it follows easily from conditions (a)–(d) above that every classical solution of (16) on $[0, \infty)$ also satisfies

(h) $\begin{bmatrix} x \\ u \end{bmatrix} \in C([0, \infty); \mathcal{D}(S))$, where the continuity is with respect to the graph norm of S on $\mathcal{D}(S)$.

Proposition 2. Let S be a system node on (U, X, Y). If $u \in C^2([0, \infty); U)$ and $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \mathcal{D}(S)$, then the equation (16) has a unique classical solution (x, u, y) satisfying $x(0) = x_0$. Moreover, this classical solution satisfies

$$x \in C^2([0, \infty); X_{-1}).$$

Well-posed system nodes

We denote by $\mathcal{D}$ the space of all the pairs $(x_0, u) \in X \times L^2([0,\infty); U)$ which satisfy the assumptions of Proposition 2. $\mathcal{D}$ is dense in $X \times L^2([0,\infty); U)$. Hence, the corresponding space $\mathcal{D}_\tau$ of pairs $(x_0, \mathbf{P}_{[0,\tau]}u)$ is dense in $X \times L^2([0,\tau]; U)$. The last proposition allows us to define the operators Σ_τ from $\mathcal{D}_\tau$ to $X \times L^2([0,\tau], Y)$ such that for any solution of (16) and for any $\tau \geq 0$,

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}. \tag{17}$$

Definition. The system node Σ_{node} is called *well-posed* if for some (hence, for every) $\tau > 0$, the operator Σ_τ from (17) has a continuous extension

$$\Sigma_\tau \in \mathcal{L}(X \times L^2([0,\tau], U), X \times L^2([0,\tau], Y)).$$

Well-posed system nodes - continued

It is easy to see that Σ_{node} is well-posed iff for some (hence, for every) $\tau > 0$ there is a $c_\tau \geq 0$ such that for all classical solutions of (16),

$$\|x(\tau)\|_X^2 + \|y\|_{L^2([0,\tau];\,Y)}^2 \leq c_\tau^2 \left(\|x(0)\|_X^2 + \|u\|_{L^2([0,\tau];\,U)}^2 \right) .$$

It is easy to verify that if Σ_{node} is well-posed, then the family $\Sigma = (\Sigma_\tau)_{\tau \geq 0}$ is a well-posed linear system. Conversely, every well-posed linear system determines a unique well-posed system node, and hence it makes sense to talk about the *combined observation/feedthrough operator* or about the *system operator of a well-posed linear system*. If Σ is a well-posed linear system with system operator S, then the dual system has the system operator S^*. (This is not trivial.)

Proposition 3. Every well-posed system node is compatible.

The Λ-extension of C and the step response

Definition. Let X and Y be Hilbert spaces, let $\mathbb{T}$ be a strongly continuous semigroup on X and let $C \in \mathcal{L}(X_1, Y)$. The Λ-*extension* of C is the operator

$$C_\Lambda x_0 \; = \; \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} x_0,$$

with its domain $\mathcal{D}(C_\Lambda)$ consisting of those $x_0 \in X$ for which the limits exist.

It is easy to see that C_Λ is indeed an extension of C.

Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ be a well-posed linear system with input space U, state space X and output space Y, with transfer function $\mathbf{G}$. The operators $A, B, C, C\&D$ are as before. χ is the characteristic function of $[0, \infty)$.
Definition. For any $\mathrm{v} \in U$, the function $y_\mathrm{v} = \mathbb{F}_\infty(\chi \cdot \mathrm{v})$ is the *step response* of Σ corresponding to v.

Regular linear systems

Definition. The system Σ is called *regular* if the following limit exists in Y, for every $\mathrm{v} \in U$:

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau y_{\mathrm{v}}(\sigma) \mathrm{d}\sigma = D\mathrm{v}. \qquad (18)$$

The operator $D \in \mathcal{L}(U, Y)$ defined by (18) is called the *feedthrough operator* of Σ.

Theorem 4. If Σ is regular, and if we denote the feedthrough operator of Σ by D, then the output y of Σ is given by

$$y(t) = C_\Lambda x(t) + Du(t), \qquad (19)$$

for almost every $t \geq 0$ (in particular, $x(t) \in \mathcal{D}(C_\Lambda)$ for almost every $t \geq 0$). If $t \geq 0$ is such that both u and y are continuous from the right at t, then (using those right limits) (19) holds at t (in particular, $x(t) \in \mathcal{D}(C_\Lambda)$).

Regular linear systems - continued

Theorem 4 implies the following formula for $\mathbb{F}_\infty$ for regular systems:

$$(\mathbb{F}_\infty u)(t) = C_\Lambda \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) \mathrm{d}\sigma + Du(t), \qquad (20)$$

for every $u \in L^2_{\mathrm{loc}}([0,\infty); U)$ and almost every $t \geq 0$ (in particular, this integral is in $\mathcal{D}(C_\Lambda)$ for almost every $t \geq 0$).

The operators A, B, C and D are called the *generating operators* of Σ. The transfer function of Σ is

$$\mathbf{G}(s) = C_\Lambda (sI - A)^{-1} B + D, \qquad \operatorname{Re} s > \omega_0(\mathbb{T})$$

(in particular, $(sI - A)^{-1} BU \subset \mathcal{D}(C_\Lambda)$).

We introduce a notation for angular domains in the complex planem: for any $\psi \in (0, \pi)$,

$$\mathcal{W}(\psi) = \left\{ re^{i\phi} \mid r \in (0,\infty),\ \phi \in (-\psi, \psi) \right\}.$$

Regular linear systems - Tauberian theorem

Theorem 5. The following statements are equivalent:

1. Σ is regular, i.e., for every $\mathrm{v} \in U$ the limit in (18) exists.
2. For every $s \in \rho(A)$ we have that $(sI - A)^{-1} BU \subset \mathcal{D}(C_\Lambda)$ and $C_\Lambda (sI - A)^{-1} B$ is an analytic $\mathcal{L}(U, Y)$-valued function of s on $\rho(A)$, uniformly bounded on any half-plane $\mathrm{Re}\, s > \omega$, where $\omega > \omega_0(\mathbb{T})$.
3. There exists $s \in \rho(A)$ such that $(sI - A)^{-1} BU \subset \mathcal{D}(C_\Lambda)$.
4. Any state trajectory of Σ is almost always in $\mathcal{D}(C_\Lambda)$.
5. For every $\mathrm{v} \in U$ and $\psi \in \left(0, \frac{\pi}{2}\right)$, $\mathbf{G}(s)\mathrm{v}$ has a limit as $|s| \to \infty$ and $s \in \mathcal{W}(\psi)$.
6. For every $\mathrm{v} \in U$, $\mathbf{G}(\lambda)\mathrm{v}$ has a limit as $\lambda \to +\infty \ \ (\lambda \in \mathbb{R})$.

Moreover, if the limits mentioned in statements (1), (5) and (6) above exist, then they are equal to $D\mathrm{v}$, where D is the feedthrough operator of Σ.

Well-posed linear systems with feedback

We now recall some static output feedback theory. We use the standing notation of this section, so that Σ is a well-posed system and U, X, Y, A, B, C, $\mathbf{G}(s)$ have their usual meaning. We take a *feedback operator* $K \in \mathcal{L}(Y, U)$ and we are interested in the *closed-loop system* Σ^K that is obtained by imposing the "static output feedback law" $u = Ky + v$, where v is the new input function, as shown in Figure 1 below. The state and output of Σ^K should be the same as for Σ, as long as their inputs are related by $u = Ky + v$. The trouble with the feedback interconnection from Figure 1 is that it is not necessarily well-posed - sometimes it cannot even be defined. To avoid such situations, we have to introduce the following concept:

Definition. $K \in \mathcal{L}(Y, U)$ is called an *admissible feedback operator* for Σ (or for $\mathbf{G}$) if $I - \mathbf{G}K$ is invertible on some right half-plane and its inverse is proper.

Here, $I - \mathbf{G}K$ may be replaced equivalently with $I - K\mathbf{G}$.

Figure 1: block diagram with feedback

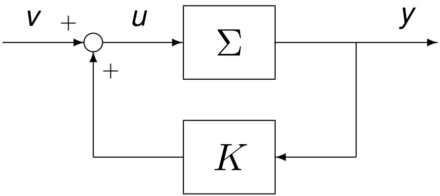

Figure: A well-posed linear system Σ with output feedback via K. If K is admissible, then this is a new well-posed linear system Σ^K, called the closed-loop system.

Static output feedback -continued

Proposition 6. If K is admissible, then the feedback connection from Figure 1 determines a new well-posed linear system $\Sigma^K = (\Sigma^K_\tau)_{\tau \geq 0}$, defined as follows: for each $\tau > 0$, Σ^K_τ is the unique solution of

$$\Sigma^K_\tau - \Sigma_\tau = \Sigma_\tau \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma^K_\tau \,. \qquad (21)$$

Moreover, the transfer function of Σ^K, denoted by $\mathbf{G}^K$, is given by

$$\mathbf{G}^K = \mathbf{G}(I - K\mathbf{G})^{-1} = (I - \mathbf{G}K)^{-1}\mathbf{G}$$

and we have the commutation property

$$\Sigma_\tau \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma^K_\tau = \Sigma^K_\tau \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma_\tau \,.$$

We denote by (A^K, B^K, C^K) the generating triple of Σ^K.

First theorem about static output feedback

Unless B is bounded, the domain $\mathcal{D}(A^K)$ may be different from $\mathcal{D}(A)$ and similarly, unless C is bounded, the space X_{-1}^K (the completion of X with respect to the norm $\|x\|_{-1}^K = \|(\beta I - A^K)^{-1}x\|$) may be different from X_{-1}.

Theorem 7. With the above notation, the following identities hold when $\mathrm{Re}\, s > \max\{\omega_0(\mathbb{T}), \omega_0(\mathbb{T}^K)\}$:

$$[I - \mathbf{G}(s)K]\, C^K(sI - A^K)^{-1} = C(sI - A)^{-1},$$

$$(sI - A^K)^{-1}B^K\,[I - K\mathbf{G}(s)] = (sI - A)^{-1}B.$$

For all $x \in \mathcal{D}(A^K)$ and for all $z \in \mathcal{D}(A)$,

$$A^K x = \left(A + BKC^K\right)x, \quad Az = \left(A^K - B^K KC\right)z,$$

where in the first formula, A is regarded as an operator from X to X_{-1}, and in the second formula, A^K is regarded as an operator from X to X_{-1}^K.

Second theorem about static output feedback

Theorem 8. With the above notation, with admissible K, assume that Σ is regular with feedthrough operator D. Then $I - DK$ (and hence also $I - KD$) is left invertible. The closed-loop system Σ^K is regular if and only if $I - DK$ (and hence also $I - KD$) is invertible. In this case, denoting the feedthrough operator of Σ^K by D^K, the operators A^K, B^K, C^K, D^K can be expressed in terms of A, B, C, D:

$$A^K x = \left[A + BK(I - DK)^{-1} C_\Lambda \right] x ,$$

$$C^K x = (I - DK)^{-1} C_\Lambda x ,$$

for all $x \in \mathcal{D}(A^K)$, where

$$\mathcal{D}(A^K) = \{ q \in \mathcal{D}(C_\Lambda) \,|\, (A + BK(I - DK)^{-1} C_\Lambda) q \in X \}.$$

Theorem 8 - continued

Moreover, we have

$$\mathcal{D}(C_\Lambda^K) = \mathcal{D}(C_\Lambda), \qquad C_\Lambda^K = (I - DK)^{-1} C_\Lambda .$$

Regarding the operators B^K and D^K we have

$$B^K = B(I - KD)^{-1},$$

$$D^K = D(I - KD)^{-1} = (I - DK)^{-1} D.$$

The motivation for introducing regular linear systems has been the simple structure of the output equation (as given in Theorem 4) and the resulting simple formula for the transfer function, as well as the more transparent feedback theory (Theorem 7), because these allow us to try to replicate classical ideas from finite-dimensional control theory in an infinite-dimensional context. Good examples of this are Luenberger observers, leading to dynamic stabilization and coprime factorization.

Passive systems with respect to a storage function

Driven by physical examples, there has been much interest in systems that are passive, which means that they satisfy some sort of energy balance inequality. To define this concept we need a function $H \in C^1(X; \mathbb{R}_+)$ called the *Hamiltonian* or *storage function*. This function is often the physical energy stored in the system, but it does not have to be. We also need a real-valued function S called the *supply rate* defined on $U \times Y$ that is usually assumed to be continuous. The system is called *passive with respect to the storage function H and the supply rate S* if for any functions u, x and y that are classical solutions of the system equations, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) \leq S(u(t), y(t)). \tag{22}$$

H is called colorred *proper* if $H(x) \to \infty$ when $\|x\| \to \infty$.

Impedance passive systems

Definition. The system node Σ_{node} on (U, X, Y) is called *impedance passive* if $Y = U'$ (the dual space of U) and all the classical solutions of (16) satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leq 2\mathrm{Re}\,\langle u(t), y(t)\rangle_{U,Y} \qquad \forall\, t \geq 0.$$

An equivalent condition is that all the generalized solutions of (16) satisfy

$$\|x(\tau)\|^2 - \|x(0)\|^2 \leq 2\int_0^\tau \mathrm{Re}\,\langle u(t), y(t)\rangle_{U,Y}\,\mathrm{d}t \qquad \forall\, \tau \geq 0.$$

Often U' is identified with U. Impedance passive systems appear frequently as models of physical systems, and then often $\frac{1}{2}\|x\|^2$ represents the energy of the system and $\mathrm{Re}\,\langle u, y\rangle$ is the power flowing into it. For example, if a component of u is a voltage (or a velocity) then the corresponding component of y is normally a current (or a force).

Staffans theorem on impedance passive systems

Theorem 9. If we identify $U' = U$, then Σ_{node} is impedance passive if and only if the operator

$$T = \begin{bmatrix} A & B \\ -C\&D \end{bmatrix}, \qquad \mathcal{D}(T) = \mathcal{D}(C\&D)$$

is dissipative (equivalently, m-dissipative) on $X \times U$. Moreover, we always have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = 2\mathrm{Re}\,\langle u(t), y(t)\rangle_U \qquad \forall\, t \geq 0$$

if and only if $\mathrm{Re}\,\langle T\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right]\rangle = 0$ for all $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \mathcal{D}(C\&D)$.

The transfer function of an impedance passive system node is always positive, i.e.,

$$\mathbf{G}(s) + \mathbf{G}^*(s) \geq 0 \quad \text{for} \quad \mathrm{Re}\, s > 0,$$

but it need not be proper. The semigroup of an impedance passive system node is always contractive.

Well-posedness for impedance passive systems

Theorem 10. (Staffans) Let Σ_{node} be an impedance passive system node. If its transfer function **G** is bounded on a vertical line in the complex plane, then Σ_{node} is well-posed.

The converse is obviously true.

Remark. There is a vaguely similar theorem about system nodes: If A is a generator, B and C are admissible and the transfer function is bounded on a vertical line, then the node is well-posed.

Scales of Hilbert spaces. Assume that H is a Hilbert space and $A_0 : \mathcal{D}(A_0) \to H$ is positive and boundedly invertible. We introduce the scale of Hilbert spaces H_α, $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geq 0$, $H_\alpha = \mathcal{D}(A_0^\alpha)$, with the norm $\|z\|_\alpha = \|A_0^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as follows: $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$.

Special structure "undamped second order"

The operator A_0 can be extended (or restricted) to each H_α, such that it becomes a bounded operator

$$A_0 : H_\alpha \to H_{\alpha-1} \qquad \forall\, \alpha \in \mathbb{R}.$$

For H and A_0 as above, let $C_0 : H_{\frac{1}{2}} \to U$. We identify U with its dual and we denote $B_0 = C_0^*$, so that $B_0 : U \to H_{-\frac{1}{2}}$. We consider the system described by

$$\ddot{z}(t) + A_0 z(t) = B_0 u(t), \tag{23}$$

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t), \tag{24}$$

where $t \in [0, \infty)$ is the time. The equation (23) is understood as an equation in $H_{-\frac{1}{2}}$. Most of the linear equations modelling the undamped vibrations of elastic structures can be written in the form (23), where z stands for the displacement field. The state $x(t)$ of this system, its state space X and its semigroup generator $A : H_1 \times H_{\frac{1}{2}} \to X$ are defined by

Systems with the special structure "undamped second order" - continued

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}, \qquad X = H_{\frac{1}{2}} \times H, \qquad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}.$$

The observation operator is $C = [0 \;\; C_0]$, defined on $\mathcal{D}(A) = X_1 = H_1 \times H_{\frac{1}{2}}$, while $B = C^*$. The operator C has a natural extension $\overline{C} : H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to U$, given by the same formula. This space $H_half \times H_{\frac{1}{2}}$ contains Z, so that the system is compatible. We can define the $\mathcal{L}(U)$-valued function $\mathbf{G}$ by

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B = s\,C_0(s^2 I + A_0)^{-1}B_0\,.$$

Define $C\&D \left[\begin{smallmatrix} x \\ v \end{smallmatrix} \right] = \overline{C}x$.

Proposition 11. $(A, B, C, \mathbf{G})$ is an impedance passive system node on (U, X, U).

Scattering passive systems

The system node S is called *scattering passive* if all the classical solutions of (16) satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leq \|u(t)\|^2 - \|y(t)\|^2 \qquad \forall\; t \geq 0\,.$$

An equivalent condition is that

$$\|x(\tau)\|^2 + \int_0^\tau \|y(t)\|^2 \mathrm{d}t \leq \|x(0)\|^2 + \int_0^\tau \|u(t)\|^2 \mathrm{d}t \qquad \forall\; t \geq 0\,.$$

The system node S is called *scattering energy preserving* if the power balance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2 \qquad \forall\; t \geq 0$$

holds for all classical solutions of (16).

Scattering passive systems - continued

The system node S is called *scattering conservative* if both S and S^* are scattering energy preserving. These systems have attracted much attention, and there are very refined tests to check if a system node is scattering conservative. Various types of Cayley transforms play an important role in the theory.

Clearly any scattering passive system is well-posed.

It is of interest to identify large classes of systems where the operators A, B, C, D have a special structure observed in models of mathematical physics, which implies that the system is scattering passive or conservative.

We shall now present a large special class of scattering passive system nodes (that includes “thin air” systems). We were led to introduce this class by our failure to fit Maxwell’s equations into the “thin air” framework. The new class of systems is also more flexible for allowing time-varying and nonlinear extensions.

The Maxwell class of scattering passive systems

We consider a linear system Σ with state space $X = H \oplus E$, where H and E are Hilbert spaces. The Hilbert space U is both the input space and the output space of Σ. The Hilbert space E_0 is a dense subspace of E and the embedding $E_0 \hookrightarrow E$ is continuous. We denote by E_0' the dual of E_0 with respect to the pivot space E, so that

$$E_0 \subset E \subset E_0',$$

densely and with continuous embeddings. We denote $X_0 = H \oplus E_0$, so that $X_0' = H \oplus E_0'$. We assume that

$$L \in \mathcal{L}(E_0, H), \qquad K \in \mathcal{L}(E_0, U), \qquad G \in \mathcal{L}(E_0, E_0'),$$

$$\mathrm{Re}\, \langle G w_0, w_0 \rangle_{E_0', E_0} \leq 0 \qquad \forall\, w_0 \in E_0,$$

and we define $\overline{A} \in \mathcal{L}(X_0, X_0')$, $B \in \mathcal{L}(U, X_0')$ and $\overline{C} \in \mathcal{L}(X_0, U)$ by

$$\overline{A} = \begin{bmatrix} 0 & -L \\ L^* & G - \frac{1}{2} K^* K \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ K^* \end{bmatrix}, \qquad \overline{C} = \begin{bmatrix} 0 & -K \end{bmatrix}.$$

The equations of the system are (as at (16))

$$\dot{x}(t) = \overline{A}x(t) + Bu(t), \qquad y(t) = \overline{C}x(t) + u(t), \tag{25}$$

where x is the state trajectory, u is the input function and y is the output function. Note that the differential equation above is an equation in X_0'. We define the domain $\mathcal{D}(A)$ by

$$\mathcal{D}(A) = \{x_0 \in X_0 \mid \overline{A}x_0 \in X\}$$

and we denote by A and C the restrictions of $\overline{A}$ and $\overline{C}$ to $\mathcal{D}(A)$. More explicitly,

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in X_0 \;\middle|\; L^* z_0 + \left(G - \tfrac{1}{2}K^*K\right) w_0 \in E \right\}.$$

We assume that $\begin{bmatrix} L \\ K \end{bmatrix}$ (with domain E_0) is closed as an unbounded operator $E \to H \oplus U$. This implies that A is maximal dissipative and hence it generates a semigroup of contractions.

Main result on the Maxwell class of systems

Theorem 12. Under the above assumptions, the equations (25) determine a scattering passive system node with state space $X = H \oplus E$ and input and output space U. This system node is scattering conservative if and only if

$$\mathrm{Re}\,\langle G w_0, w_0 \rangle_{E_0', E_0} = 0 \qquad \forall\, w_0 \in E_0 .$$

The operator of this syste node is

$$S_{\mathrm{sca}} = \begin{bmatrix} [A\&B]_{\mathrm{sca}} \\ [C\&D]_{\mathrm{sca}} \end{bmatrix}$$

where

$$[A\&B]_{\mathrm{sca}} = \begin{bmatrix} 0 & -L & 0 \\ L^* & G - \frac{1}{2}K^*K & K^* \end{bmatrix}, \quad [C\&D]_{\mathrm{sca}} = \begin{bmatrix} 0 & -K & I \end{bmatrix}$$

both have the domain $\mathcal{D}(S_{\mathrm{sca}})$ given by

$$\left\{ \begin{bmatrix} z_0 \\ w_0 \\ u_0 \end{bmatrix} \in H \times E_0 \times U \;\middle|\; L^* z_0 + \left(G - \tfrac{1}{2}K^*K\right) w_0 + K^* u_0 \in E \right\}.$$

Proposition about classical solutions

We use the notation and the assumptions of the main result. If the input function u and the initial state $\begin{bmatrix} z(0) \\ w(0) \end{bmatrix}$ of $S_{\rm sca}$ satisfy

$$u \in \mathcal{H}^1_{\rm loc}((0,\infty);U)\,, \qquad \begin{bmatrix} z(0) \\ w(0) \\ u(0) \end{bmatrix} \in \mathcal{D}(S_{\rm sca})\,, \tag{26}$$

then the corresponding state trajectory $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ and output function y of $S_{\rm sca}$ satisfy $y \in \mathcal{H}^1_{\rm loc}((0,\infty);Y)$,

$$\begin{bmatrix} z \\ w \end{bmatrix} \in C^1([0,\infty);H \oplus E)\,, \quad \begin{bmatrix} z \\ w \\ u \end{bmatrix} \in C([0,\infty);\mathcal{D}(S_{\rm sca}))\,,$$

and

$$\begin{bmatrix} \dot z(t) \\ \dot w(t) \\ y(t) \end{bmatrix} = S_{\rm sca} \begin{bmatrix} z(t) \\ w(t) \\ u(t) \end{bmatrix} \qquad \forall\, t > 0\,.$$

Power balance formula

We use the notation and the assumptions of the main result. If the functions $u, x = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ and y are a classical solution of

$$\begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \\ y(t) \end{bmatrix} = S_{\mathrm{sca}} \begin{bmatrix} z(t) \\ w(t) \\ u(t) \end{bmatrix} \qquad \forall\, t > 0\,,$$

then the following power balance equation holds for $t \geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2 + 2\mathrm{Re}\,\langle Gw(t), w(t)\rangle\,.$$

The dual system node S^*_{sca} has the same structure, but with L, K and G replaced with $-L$, $-K$ and G^*. Therefore, its classical solutions satisfy the same power balance equation. (Hence, as already mentioned, S_{sca} is scattering conservative if and only if $\mathrm{Re}\,\langle Gw_0, w_0\rangle = 0$ for all $w_0 \in E_0$.)

The generating operators

We use the notation and the assumptions of the main result. We have

$$X_1 \subset X_0 \subset X \subset X_0' \subset X_{-1},$$

where all the embeddings are continuous and dense. A (the semigroup generator of $S_{\rm sca}$) has a unique extension to an operator $A \in \mathcal{L}(X, X_{-1})$, whose restriction to X_0 is $\overline{A}$.

The control operator B of $S_{\rm sca}$ is $\left[\begin{smallmatrix} 0 \\ K^* \end{smallmatrix}\right]$ and the observation operator C of $S_{\rm sca}$ is $\overline{C} = [0 \ \ -K]$ restricted to $\mathcal{D}(A)$. The transfer function of $S_{\rm sca}$ is

$$\mathbf{G}(s) = I - K\left[sI + \frac{1}{2}K^*K - G + \frac{1}{s}L^*L\right]^{-1} K^*,$$

for all s in the open right half-plane.

The hidden variable

If $S_{\rm sca}$ is a system node of the above structure, and moreover L is onto, then for any classical solution $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right], u, y$ of (25) it is possible to identify a *hidden variable* q such that

$$z(t) = -Lq(t), \qquad w(t) = \dot{q}(t).$$

Indeed, first we have to find an initial value for q such that $z(0) = -Lq(0)$ (this $q(0)$ may be non-unique). Then, for every $t \geq 0$ put $q(t) = q(0) + \int_0^t w(\sigma)\mathrm{d}\sigma$. After introducing q, the expression $V(z(t)) = \frac{1}{2}\|z(t)\|^2 = \frac{1}{2}\|Lq(t)\|^2$ can be interpreted as the *potential energy* stored in Σ, while $\frac{1}{2}\|w(t)\|^2 = \frac{1}{2}\|\dot{q}(t)\|^2$ can be interpreted as the *kinetic energy*. The hidden variable q satisfies the differential equation

$$\ddot{q}(t) + (\tfrac{1}{2}K^*K - G)\dot{q}(t) + L^*Lq(t) = K^*u(t),$$

where all the terms are in E_0', $q \in C^1([0,\infty); E_0)$ and $q \in C^2([0,\infty); E)$.

The difference between two different hidden variables associated to the same classical solution of (25) is a constant in $\operatorname{Ker} L$. An alternative state for this system is $(q, \dot{q})$, taking values in the new state space $E_0 \oplus E$ (this is often used in the literature). Note that, since L is not assumed to be closed, in general it is not possible to restrict $L^* L$ to a positive (in particular, self-adjoint) operator from $\mathcal{D}(L^* L) \subset E$ to E.

Systems described by such second order operator ODEs have been studied by many authors, we mention H.O. Fattorini, B. Jacob, K. Morris, Section VI.3 in the book of Engel and Nagel.

An interesting generalization of the class of systems discussed here arises if, instead of the quadratic expression for the potential energy $V(z(t)) = \frac{1}{2}\|z(t)\|^2$, we allow it to be a more general nonlinear C^1 function of $z(t)$, denoted V. Here, we assume that all the Hilbert spaces are real, which is more realistic for non-linear systems. The energy in the system is defined by $\mathcal{E}(t) = V(z(t)) + \frac{1}{2}\|w(t)\|^2$.

The equations of the system are postulated to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 0 & -L \\ L^* & G - \frac{1}{2}K^*K \end{bmatrix} \begin{bmatrix} (\nabla V)(z(t)) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K^* \end{bmatrix} u(t),$$

$$y(t) = -Kw(t) + u(t),$$

with the same assumptions on L, K, G as before. These are particular cases of scattering versions of *port-Hamiltonian systems*, studied by A. van der Schaft, B. Maschke and others. For every solution of the above equations we have the power balance equation

$$2\dot{\mathcal{E}}(t) = \|u(t)\|^2 - \|y(t)\|^2 + 2\langle Gw(t), w(t)\rangle \qquad \forall\, t \geq 0.$$

For most systems in this class, it is challanging to prove the existence and uniqueness of suitable solutions of the system equations for a dense set of initial states and input functions.

Impedance passive systems

An *impedance passive (or conservative)* system node has equal input and output spaces and the solutions of (16)) satisfy

$$\|x(\tau)\|^2 - \|x(0)\|^2 \leq 2\int_0^\tau \mathrm{Re}\,\langle e(t), f(t)\rangle\,\mathrm{d}t \qquad \forall\,\tau \in [0, T)$$

(or the corresponding equality). Here, we have denoted the input signal by e (sometimes called the effort) and the output signal by f (sometimes called the flow), and of course (16) should be written with these signals in place of u and y.

It is always possible to transform an impedance passive or conservative system node $S_{\rm imp}$ into a scattering passive or conservative system node $S_{\rm sca}$ by the *external Cayley transformation* which redefines the input and the output as follows:

$$u = \tfrac{1}{\sqrt{2}}(e+f), \qquad y = \tfrac{1}{\sqrt{2}}(e-f).$$

The inverse transformation is given by the same formulas, only with the places of u, y and e, f reversed, as is easy to see.

The external Cayley transformation can be understood also as a feedback transformation, as Figure 2 shows. It is easy to see from this figure that the relation between the transfer functions of $S_{\rm imp}$ and $S_{\rm sca}$ is

$$\mathbf{G}_{\rm sca} = (I - \mathbf{G}_{\rm imp})(I + \mathbf{G}_{\rm imp})^{-1}\,.$$

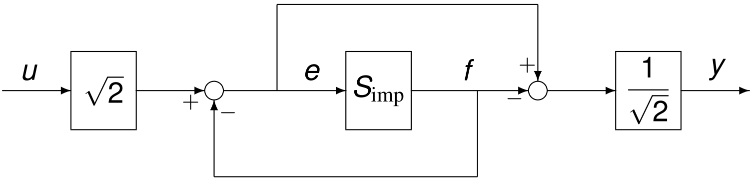

Figure: The system node $S_{\rm sca}$ with input u and output y, as obtained from the system node $S_{\rm imp}$ (with input e and output f).

The actual formulas linking $S_{\rm imp}$ and $S_{\rm sca}$ are complicated and we omit writing them here.

The main abstract result

Theorem 12. Let $S_{\mathrm{imp}} = \begin{bmatrix} [A\&B]_{\mathrm{imp}} \\ [C\&D]_{\mathrm{imp}} \end{bmatrix}$ be an operator mapping its domain $\mathcal{D}(S_{\mathrm{imp}}) \subset X \oplus U$ into $X \oplus U$, such that $T := \begin{bmatrix} [A\&B]_{\mathrm{imp}} \\ -[C\&D]_{\mathrm{imp}} \end{bmatrix}$ (with the same domain) is maximal dissipative. We define S_{sca} via the external Cayley transform applied to S_{imp}. Then S_{sca} is a scattering passive system node. The system node S_{sca} is scattering conservative if and only if T is skew-adjoint.

To get our result on the Maxwell class of well-posed systems, we apply the above theorem for

$$T = \begin{bmatrix} 0 & -L & 0 \\ L^* & G & K_0^* \\ 0 & -K_0 & 0 \end{bmatrix},$$

which is maximal dissipative by a relatively simple argument.