Minimal order controllers for nonlinear output regulation

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Informal problem statement

Informal statement: Given a stabilizable plant and a totally unstable exosystem, the

error feedback regulator problem

is to design a stabilizing controller (for the plant) which guarantees the tracking of certain reference signals by the plant output, even when the plant is driven by external disturbance signals. The reference and disturbance signals are both functions of the exosystem state. The input of the controller is the tracking error and its output is the control input to the plant.

Best known example: for linear systems, under suitable conditions, the PI controller solves the error feedback regulator problem for an exosystem generating constant reference and disturbance signals. If we are lucky, it works even for nonlinear systems - we hope to make this more clear in the sequel.

The plant and the exosystem

To make a rigorous statement of the error feedback regulator problem (one of the many possible versions), we consider a nonlinear finite-dimensional smooth *plant*

$$\dot{x} = f(x, u, w), \qquad y = g(x, u, w), \tag{1}$$

with state $x(t) \in X \subset \mathbb{R}^n$, control input $u(t) \in U \subset \mathbb{R}$ and output $y(t) \in Y \subset \mathbb{R}$, where X, U and Y are open sets that contain the origin of the appropriate spaces. The exogenous disturbance signal w in (1) is the state of the linear *exosystem*

$$\dot{w} = Sw, \tag{2}$$

with $w(t) \in W \subset \mathbb{R}^{2p+1}$, where W is open, invariant under e^{St} ($t \geq 0$) and $0 \in W$. We assume that the functions $f : X \times U \times W \to \mathbb{R}^n$ and $g : X \times U \times W \to Y$ are of class C^2 and $f(0, 0, 0) = 0$, $g(0, 0, 0) = 0$.

The plant and the exosystem - continued

$$S = \begin{bmatrix} \alpha_0 & 0 & 0 & \cdots & 0 \\ 0 & S_1 & 0 & \dots & 0 \\ 0 & 0 & S_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & S_p \end{bmatrix}, \qquad \alpha_0 = 0,$$

$$S_j = \begin{bmatrix} 0 & \alpha_j \\ -\alpha_j & 0 \end{bmatrix}, \quad \alpha_j > 0 \qquad \forall\, j \in \{1, 2, \dots p\}.$$

Hence the exosystem can generate a constant and sinusoids. The *reference signal* is $y_r = q(w)$, where $q : W \to Y$ is assumed to be of class C^2 with $q(0) = 0$. The *tracking error* is

$$e(t) = y(t) - y_r(t) = g(x, u, w) - q(w) = h(x, u, w). \qquad (3)$$

Clearly $h(x, u, w) : X \times U \times W \to Y_e$ is a C^2 map satisfying $h(0, 0, 0) = 0$. Here $Y_e \subset \mathbb{R}$ is open and $Y \subset Y_e$.

The linearization of the plant

We define the real matrices

$$A = \left[\frac{\partial f}{\partial x}\right]_{(0,0,0)}, \quad B = \left[\frac{\partial f}{\partial u}\right]_{(0,0,0)}, \quad P = \left[\frac{\partial f}{\partial w}\right]_{(0,0,0)},$$

$$C = \left[\frac{\partial h}{\partial x}\right]_{(0,0,0)}, \quad D = \left[\frac{\partial h}{\partial u}\right]_{(0,0,0)}, \quad Q = \left[\frac{\partial h}{\partial w}\right]_{(0,0,0)}.$$

We will need the following linearization of the plant (1) and the error (3) (linearized at $(x, u, w) = (0, 0, 0)$):

$$\dot{x} = Ax + Bu + Pw, \tag{4}$$

$$e_l = Cx + Du + Qw. \tag{5}$$

The transfer function of the linear system (4), (5) from the control input u to the linearized error e_l is

$$\mathbf{G}(s) = C(sI - A)^{-1}B + D, \qquad s \in \rho(A).$$

The controller

Consider the controller

$$\dot{\xi} = \eta(\xi, e), \qquad u = \theta(\xi), \tag{6}$$

with state $\xi(t) \in X_c \subset \mathbb{R}^{n_c}$, input e and output u, where X_c is open and contains 0. The functions $\eta : X_c \times Y_e \to \mathbb{R}^{n_c}$ and $\theta : X_c \to U$ are C^2 maps satisfying $\eta(0,0) = 0$ and $\theta(0) = 0$. The closed-loop system consisting of the plant (1), the exosystem (2) and the controller (6) will be shown in a figure a little later. We define the matrices

$$F = \left[\frac{\partial \eta}{\partial \xi}\right]_{(0,0)}, \quad G = \left[\frac{\partial \eta}{\partial e}\right]_{(0,0)}, \quad K = \left[\frac{\partial \theta}{\partial \xi}\right]_{0}, \tag{7}$$

which determine the linearization of the controller. The *order* of this controller is, by definition, n_c.

More on the controller, and an assumption

Thus, the linearized controller is described by

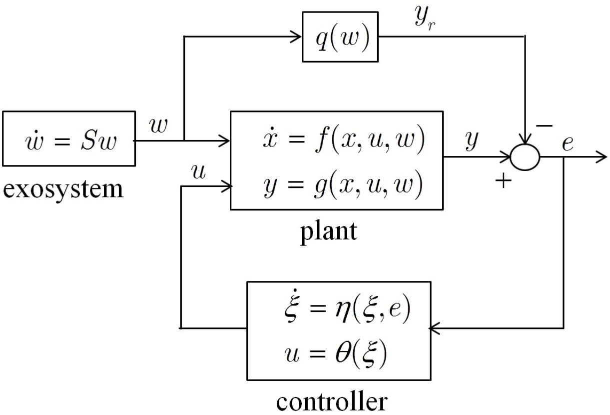
$$\dot{\xi} = F\xi + Ge, \qquad u = K\xi.$$

Assumption. The matrix A is stable (i.e., $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \sigma(A)$) and the pair of matrices $\left(\begin{bmatrix} C & Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \right)$ is detectable.

The above detectability assumption tells us that, assuming $u = 0$, we can see the entire exosystem state from the error signal (a nonzero exosystem state cannot cause a zero error). In the linear case, if the above detectability assumption does not hold, then we can simply replace the exosystem with a smaller one (we throw out the non-observable eigenvalues of S). In the nonlinear case this is not always possible.

If A is not stable, then we have to perform a preliminary stabilization step. The stability assumption on A can be relaxed, as we shall see later.

The error feedback regulator problem



This diagram shows the closed-loop system of the plant and the controller, driven by the exosystem S. The problem is to ensure that the closed loop system with $w = 0$ is locally exponentially stable and for small enough initial states, $e(t) \to 0$.

Formal problem statement

Definition. The controller (6) is said to solve the *local error feedback regulator problem* for the plant (1), the exosystem (2) and the error (3) if:

1. The equilibrium $(x, \xi) = (0, 0)$ of the unforced closed-loop system
$$\dot{x} = f(x, \theta(\xi), 0), \qquad \dot{\xi} = \eta(\xi, h(x, \theta(\xi), 0)),$$
is locally exponentially stable.
2. The forced closed-loop system
$$\dot{x} = f(x, \theta(\xi), w), \quad \dot{w} = Sw, \quad \dot{\xi} = \eta(\xi, h(x, \theta(\xi), w)), \tag{8}$$
is such that for each initial condition $(x(0), \xi(0), w(0))$ in a neighborhood of $(0, 0, 0)$ in $X \times X_c \times W$, it has a unique global (in time $t \geq 0$) solution (x, ξ, w) and the corresponding error e defined in (3) satisfies
$$\lim_{t \to +\infty} e(t) = \lim_{t \to +\infty} h(x(t), \theta(\xi(t)), w(t)) = 0. \tag{9}$$

Background and history

The regulator problem, and its robust version, for linear finite-dimensional plants and exosystems was addressed in Francis and Wonham (1975) and Francis (1977) using geometric methods. The solvability of the problem was characterized in terms of the solvability of certain matrix equations, known in the literature as *the regulator equations*. The *internal model principle*, which states that for robust regulation the dynamic structure of the exosystem (suitably duplicated) must be incorporated into the controller, was introduced in Francis and Wonham (1975).

The regulator problem for nonlinear finite-dimensional plants and exosystems was addressed in Isidori and Byrnes (1990) in a local setting, i.e., the proposed controller ensured that its closed-loop system with the plant is locally exponentially stable and that tracking is achieved for sufficiently small initial conditions of the plant, the controller and the exosystem.

Background and history - continued

In Isidori and Byrnes (1990) the nonlinear regulator equations, a generalization of the regulator equations in Francis (1977), were introduced along with the notion of zero dynamics to describe the solvability of the nonlinear regulator equations. Since the early 1990's many researchers have extended these results, mainly by developing controllers that solve the nonlinear robust regulator problem in local setting, e.g. Priscoli (1993, 1997), Byrnes, Priscoli, Isidori, Kang (1997), in the semi-global setting, e.g. Khalil (1994), Isidori (1997), Serrani, Isidori and Marconi (2000), and in the global setting, e.g. Serrani and Isidori (2000), Chen and Huang (2005), Xi and Ding (2007).

The above works assume that the exosystem is perfectly known; nonlinear regulator problems with an uncertain exosystem have been considered in Isidori, Marconi and Praly (2012), Li and Khalil (2012) etc. There is a large literature for the linear infinite-dimensional regulator problem.

A classical result of Isidori and Byrnes (1990)

Theorem. Let the pair (A, B) be stabilizable and let the detectability condition in the earlier Assumption hold. Then there exists a controller of the form (6) that solves the local error feedback regulator problem if and only if there exist an open set $W^o \subset W$ containing zero and C^2 maps $\pi : W^o \to X$ and $\gamma : W^o \to U$, with $\pi(0) = 0$ and $\gamma(0) = 0$, satisfying the nonlinear *regulator equations*

$$\frac{\partial \pi}{\partial w} S w = f(\pi(w), \gamma(w), w), \tag{10}$$

$$h(\pi(w), \gamma(w), w) = 0. \tag{11}$$

Moreover, in this case there exist open sets $Z \subset X \times X_c$ and $W^{oo} \subset W^o$, both containing zero, such that for any $(x_0, \xi_0) \in Z$ and any $w_0 \in W^{oo}$, the closed-loop system (8) has a global (in time $t \geq 0$) solution (x, ξ, w) with $x(0) = x_0$, $\xi(0) = \xi_0$, $w(0) = w_0$ and

$$\lim_{t \to \infty} \|x(t) - \pi(w(t))\| = 0, \qquad \lim_{t \to \infty} \|u(t) - \gamma(w(t))\| = 0.$$

Comments on the theorem

The standard reference for this result is the book of Isidori (1995). The function g in (1) is considered there to be independent of u. Earlier references do not present a rigorous assessment of the smoothness of the maps π and γ. Under the assumption that the plant and controller functions are of class C^2, we have shown that the maps π and γ are also of class C^2. Many intermediate results are of high interest.

The linearized version of the regulator equations is

$$\Pi S = A\Pi + B\Gamma + P, \qquad C\Pi + D\Gamma + Q = 0,$$

where

$$\Pi = \left[\frac{\partial \pi}{\partial w}\right]_{w=0}, \qquad \Gamma = \left[\frac{\partial \gamma}{\partial w}\right]_{w=0}.$$

The control law $u(t) = \Gamma w(t)$ solves the linear state feedback regulator problem for the linear plant (4), the exosystem (2) and the linearized error $e_l(t)$ from (5). This means that if $u(t) = \Gamma w(t)$, then $e_l(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $x(0)$ and $w(0)$.

Our aims

In this paper we focus on *finding a minimal order controller that solves the nonlinear error feedback regulator problem in a local setting*. Specifically, we will construct controllers whose order is the same as that of the exosystem. The order of the controllers in the papers mentioned earlier, even in the absence of uncertain plant parameters, is typically equal to or larger than that of the exosystem and the plant combined. A key reason for this is the sequential control design approach they adopt: First an internal model (whose order can be larger than that of the exosystem) is designed and then the loop containing the internal model and the plant is stabilized using an additional controller. In contrast we will design a stabilizing internal model directly. The search for minimal order controllers is of practical value from an implementation standpoint. For a discussion on the lower bound for the order of any controller that solves the linear robust regulator problem see Davison and Goldenberg (1975), Desoer and Wang (1978).

Our aims - continued

Our work is motivated by an alternate approach to the linear regulator problem, first proposed in Davison (1976) for stable finite-dimensional linear plants. In this approach, the control and observation operators of the internal model (whose order is $r \times m$, where r is the number of outputs and m is the exosystem order) are chosen to ensure that its closed-loop system with the plant is stable. This approach was extended in Hämäläinen and Pohjolainen (2000), Rebarber and Weiss (2003) to construct finite-dimensional controllers that solve the regulator problem for stable linear infinite-dimensional plants.

The Davison controller is appealing because *you do not have to solve the regulator equations* and only little information about the plant is needed (the direction of $\mathbf{G}(i\alpha_j)$ for $0 \leq j \leq p$). We will extend this approach (the Davison controller) to nonlinear finite-dimensional plants that are SISO from control input to output. Unfortunately, in the nonlinear case, we must solve the regulator equations, which in most cases are PDEs.

Restating an old result on the linear regulator problem

Remember that $\mathbf{G}(s) = C(sI - A)^{-1}B + D$ is the transfer function of the linearized plant from u to e.

Proposition. Assume that A is stable and for each $j \in \{0, 1, \ldots p\}$, $\mathbf{G}(i\alpha_j) \neq 0$. Then, there exist vectors B_c, $C_c^\top \in \mathbb{R}^{2p+1}$ such that the linear controller

$$\dot{z}_c = Sz_c + B_c e_l, \qquad u = C_c z_c, \tag{12}$$

for the linearized plant (4), (5) renders the closed-loop system stable, i.e.,

$$\mathcal{A} = \begin{bmatrix} A & BC_c \\ B_c C & S + B_c D C_c \end{bmatrix} \tag{13}$$

is a stable matrix. For any such B_c, C_c, the controller in (12) solves the linear error feedback regulator problem for the linearized plant (4), the exosystem (2) and the linearized error (5), and moreover in this case the linearized error $e_l(t)$ converges to zero exponentially.

Comments on the last proposition

The above proposition can be derived easily from Theorem 1.1 in Rebarber and Weiss (2003), which tells us also how to find B_c and C_c: Choose $K_j \in \mathbb{C}$ with $\mathrm{Re}\,[\mathbf{G}(i\alpha_j)K_j] > 0$ for $j \in \{0, 1, \ldots p\}$, and such that $K_0 \in \mathbb{R}$. Denote

$$B_c = [b_0, b_1, \ldots b_{2p}]^\top, \qquad C_c = [c_0, c_1, \ldots c_{2p}].$$

Then $\mathcal{A}$ will be stable if

$$c_0 b_0 = -\varepsilon K_0 \quad \text{and} \quad (c_{2j-1} + ic_{2j})(b_{2j-1} - ib_{2j}) = -2\varepsilon K_j \quad (14)$$

for $1 \leq j \leq p$, where $\varepsilon > 0$ is sufficiently small.

The transfer function of the above controller is

$$\mathbf{C}(s) = -\varepsilon\left[\frac{K_0}{s} + 2\sum_{j=1}^{p} \frac{(\mathrm{Re}\,K_j)s - (\mathrm{Im}\,K_j)\alpha_j}{s^2 + \alpha_j^2}\right].$$

Our main result

Theorem. Let the earlier Assumption hold. Suppose that there exist C^2 maps $\pi : W^o \to X$ and $\gamma : W^o \to U$, where W^o is an open neighborhood of zero in W, that satisfy the nonlinear regulator equations (10) and (11), with $\pi(0) = \gamma(0) = 0$. Then $\mathbf{G}(i\alpha_j) \neq 0$ for all $j \in \{0, 1, \ldots p\}$ and there exist $B_c \in \mathbb{R}^{2p+1}$ and a possibly nonlinear C^2 map $\theta : X_c \to U$, where X_c is an open neighborhood of zero in W, such that $\theta(0) = 0$ and the controller

$$\dot{\xi} = S\xi + B_c e, \qquad u = \theta(\xi), \tag{15}$$

solves the local error feedback regulator problem for the plant (1), the exosystem (2) and the error (3). Moreover, this controller is minimal, i.e., it is of the lowest possible order among all the controllers of the form (6) that solve this error feedback regulator problem.

The vector B_c can be chosen such that

$$\lim_{t \to \infty} \|\xi(t) - w(t)\| = 0, \qquad \theta = \gamma.$$

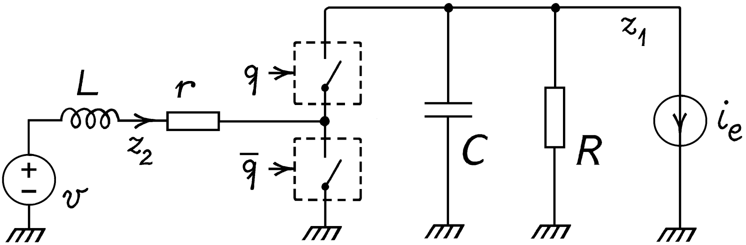
Comments on the main result

Remark. The transfer function of the controller in (12) has poles at $\pm i\alpha_j$ ($j \in \{0, 1, \dots p\}$). The necessity of the condition $\mathbf{G}(i\alpha_j) \neq 0$ for the solvability of the local error feedback regulator problem can be seen also from the following fact: in a stable closed-loop system there cannot be an unstable pole-zero cancelation in the product of the plant and controller transfer functions. This is well-known but not easy to find in the literature, see for instance Anderson and Gevers (1981).

Remark. We have assumed that the state operator A of the linearized plant is stable. This assumption is used only to guarantee the existence of vectors $B_c, C_c^\top \in \mathbb{R}^{2p+1}$ such that $\mathcal{A}$ in (13) is stable. Therefore, instead of the stability of A, we could have directly assumed the existence of such vectors. The latter assumption is more general, as can be seen on very simple examples involving 2x2 matrices.

A example: control of a boost converter

We consider the output voltage regulation for the boost power converter shown in the figure. A constant but unknown input voltage $v > 0$ is transformed into a higher voltage z_1 that feeds a load R. Due to the fast switching, there will be high frequency ripple on z_1, which becomes negligible for very high switching frequency, and we neglect this ripple. The control problem is to make z_1 track a reference value, in spite of the sinusoidal disturbance current i_e. The controller generates the high frequency binary signal q with duty cycle $\mathcal{D} \in [0, 1]$.



The operating point of the boost converter

The state variables z_1, z_2 and the inputs v and i_e are considered practically equal to their short-time averaged values. It is easy to derive the equations corresponding to the averaged variables:

$$C\dot{z}_1 = -i_e - \frac{z_1}{R} + \mathcal{D}z_2\,, \qquad L\dot{z}_2 = -rz_2 + v - \mathcal{D}z_1\,. \qquad (16)$$

We consider an operating point (an equilibrium state) corresponding to the inputs $v = v_0 > 0$, $i_e = 0$, $\mathcal{D} = \mathcal{D}_0 \in (0, 1)$. The corresponding equilibrium state $[z_{10}\ z_{20}]^\top$ can be computed by setting $\dot{z}_1 = 0$ and $\dot{z}_2 = 0$ in (16), which leads to

$$z_{10} = \frac{\mathcal{D}_0}{\frac{r}{R} + \mathcal{D}_0^2}\, v_0\,, \qquad z_{20} = \frac{\frac{1}{R}}{\frac{r}{R} + \mathcal{D}_0^2}\, v_0\,.$$

It is assumed that this is a desirable equilibrium point, i.e., z_{10} is exactly the reference output voltage.

The boost converter - rewriting it into our framework

The input voltage v may deviate from v_0 (for instance, batteries running on low charge), but this deviation is considered to be a constant at the time scale of interest. The disturbance current i_0 is considered to be sinusoidal, with known frequency $\alpha > 0$. (For instance, this disturbance might be caused by a single-phase DC/AC power converter delivering current to an AC load, in which case α would be twice the grid frequency.) The state and input variables will be the deviations of the original variables from their values at the operating point:

$$x_1 = z_1 - z_{10}\,, \;\; x_2 = z_2 - z_{20}\,, \;\; w_1 = v - v_0\,, \;\; u = \mathcal{D} - \mathcal{D}_0\,.$$

Using (16), and the notation $w_2 = i_e$, the deviations satisfy the equations:

$$\dot{x}_1 = -\frac{x_1}{RC} + \frac{\mathcal{D}_0 + u}{C} x_2 + \frac{z_{20}}{C} u - \frac{1}{C} w_2\,,$$

$$\dot{x}_2 = -\frac{\mathcal{D}_0 + u}{L} x_1 - \frac{r}{L} x_2 - \frac{z_{10}}{L} u + \frac{1}{L} w_1\,.$$

The error feedback regulator problem for the boost

The disturbance signal w_1 is an unknown constant while $w_2(t) = a\cos(\alpha t + \phi)$, where a and ϕ are unknown. We assume that w_1 and w_2 are generated by the exosystem

$$\dot{w} = Sw\,, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & -\alpha & 0 \end{bmatrix}. \tag{17}$$

We want to regulate x_1 to zero and therefore the error is

$$e = x_1\,. \tag{18}$$

It is well known that it is difficult to control the higher voltage in a boost converter because of the unstable zero dynamics (which can be seen from the presence of a right-half plane zero in the transfer function of the linearization from u to x_1).

Denote $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$. We can rewrite this system in the form of (1) and (3), by defining the appropriate functions f and h.

The boost converter - verifying the assumptions

The linearization of the plant and the error around $(0,0,0)$ is as in (4)–(5), with

$$A = \begin{bmatrix} -\frac{1}{RC} & \frac{\mathcal{D}_0}{C} \\ -\frac{\mathcal{D}_0}{L} & -\frac{r}{L} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{z_{20}}{C} \\ -\frac{z_{10}}{L} \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & -\frac{1}{C} & 0 \\ \frac{1}{L} & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0, \qquad Q = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see (from trace $A < 0$ and det $A > 0$) that A is stable. The detectability assumption contained in our assumption can be verified (for any combination of parameter values) using the Hautus test.

The nonlinear regulator equations (10), with the notation $\pi = \begin{bmatrix} \pi^1 \\ \pi^2 \end{bmatrix}$, are

$$\alpha \frac{\partial \pi^1}{\partial w_2} w_3 - \alpha \frac{\partial \pi^1}{\partial w_3} w_2 = -\frac{\pi^1}{RC} + \frac{\mathcal{D}_0 + \gamma}{C} \pi^2 + \frac{z_{20}}{C} \gamma - \frac{1}{C} w_2,$$

$$\alpha \frac{\partial \pi^2}{\partial w_2} w_3 - \alpha \frac{\partial \pi^2}{\partial w_3} w_2 = -\frac{\mathcal{D}_0 + \gamma}{L} \pi^1 - \frac{r}{L} \pi^2 - \frac{z_{10}}{L} \gamma + \frac{1}{L} w_1.$$

The regulator equations reduced to a PDE

The regulator equation (11) is $\pi^1 = 0$. Using this, we rewrite the above as

$$0 = \frac{\mathcal{D}_0 + \gamma}{C}\pi^2 + \frac{z_{20}}{C}\gamma - \frac{1}{C}w_2\,, \tag{19}$$

$$\alpha\frac{\partial\pi^2}{\partial w_2}w_3 - \alpha\frac{\partial\pi^2}{\partial w_3}w_2 = -\frac{r}{L}\pi^2 - \frac{z_{10}}{L}\gamma + \frac{1}{L}w_1\,. \tag{20}$$

Substituting for γ from (20) into (19) we get

$$\begin{aligned}&\left(\mathcal{D}_0 + \frac{L}{z_{10}}\left[-\alpha\frac{\partial\pi^2}{\partial w_2}w_3 + \alpha\frac{\partial\pi^2}{\partial w_3}w_2 - \frac{r}{L}\pi^2 + \frac{1}{L}w_1\right]\right)\pi^2\\ &+ z_{20}\frac{L}{z_{10}}\left[-\alpha\frac{\partial\pi^2}{\partial w_2}w_3 + \alpha\frac{\partial\pi^2}{\partial w_3}w_2 - \frac{r}{L}\pi^2 + \frac{1}{L}w_1\right] - w_2 = 0.\end{aligned} \tag{21}$$

This is a first order quasilinear PDE in the unknown function $\pi^2 : \mathbb{R}^3 \to \mathbb{R}$, with no boundary conditions, only a one-point condition: $\pi^2(0) = 0$.

Solving the PDE

We will solve the PDE (21) in a neighborhood of $0 \in \mathbb{R}^3$. Our approach is to determine π^2 on circles of the following type: w_1 is constant, $w_2 = \rho \cos \tau$ and $w_3 = \rho \sin \tau$, where $\rho > 0$ is a constant and $\tau \in [0, 2\pi)$. The motivation for our approach is that on such circles, the PDE (21) becomes an ODE. (In fact, these circles are the projections of the characteristic curves of (21) in $\mathbb{R}^4$, onto the space $\mathbb{R}^3$ with coordinates w_1, w_2 and w_3.) For each fixed w_1 and ρ, we define a function ψ on $[0, 2\pi)$ by

$$\psi(\tau) = \pi^2(w_1, \rho \cos \tau, \rho \sin \tau). \tag{22}$$

It follows by the chain rule that

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = -\frac{\partial \pi^2}{\partial w_2} \rho \sin \tau + \frac{\partial \pi^2}{\partial w_3} \rho \cos \tau = -\frac{\partial \pi^2}{\partial w_2} w_3 + \frac{\partial \pi^2}{\partial w_3} w_2 .$$

Solving the PDE - continued

Using the previous expression, (21) can be rewritten as

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \frac{r\psi^2 + (rz_{20} - w_1 - \mathcal{D}_0 z_{10})\psi - z_{20}w_1 + z_{10}\rho\cos\tau}{\alpha L(\psi + z_{20})}. \quad (23)$$

This ODE has to be solved for ψ that satisfies the periodic boundary condition $\psi(0) = \psi(2\pi)$. From the function ψ, corresponding to different values of w_1 and ρ, we will then construct the function π^2 using (22).

For details of how the solutions look, see our paper. Important point: we can approximate the solution WITHOUT SOLVING THE PDE by using several additional harmonics in the internal model. This is a bit like repetitive control. Then, each oscillator in the internal model will learn by itself what it has to do, like in the linear case, and it will adapt itself to a changing plant.

THANK YOU!