Internal model based tracking and disturbance rejection for stable well-posed systems

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The plant and the controller

We assume that the plant Σ_p is a well-posed linear system and it is exponentially stable. The plant has two inputs, w and u. The input w consists of the external signals (references and disturbances) and u is the control input. These signals take values in the Hilbert spaces W and U, respectively. The output signal of Σ_p, denoted by z, which represents the tracking error, takes values in the Hilbert space Y. The transfer function of the plant is

$$\mathbf{P} = [\mathbf{P}_1 \;\; \mathbf{P}_2],$$

where $\mathbf{P}_1(s) \in \mathcal{L}(W, Y)$ and $\mathbf{P}_2(s) \in \mathcal{L}(U, Y)$. Let Σ_c be the well-posed controller which is to be determined. The closed-loop system obtained by interconnecting these systems is shown in Figure 1.

Block diagram - Figure 1

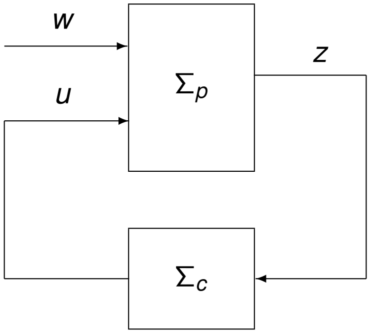

Figure: The closed-loop system built by interconnecting the stable well-posed plant Σ_p with the well-posed controller Σ_c. The signal w contains disturbances and references, and z is the error signal which should be made small. The transfer functions of Σ_p and Σ_c are $\mathbf{P} = [\mathbf{P}_1 \ \ \mathbf{P}_2]$ and $\mathbf{C}$.

The abstract description of the plant

We denote the state space of Σ_p by X (this is a Hilbert space). The state trajectories x of this system satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + B_1 w(t) + B_2 u(t), \qquad (1)$$

where A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X and B_1 and B_2 are admissible control operators for $\mathbb{T}$. This follows from the general representation theory of well-posed linear systems, which was discussed earlier. If Σ_p is regular, then z (the output function of Σ_p) is given for almost every $t \geq 0$ by

$$z(t) = C_\Lambda x(t) + D_1 w(t) + D_2 u(t), \qquad (2)$$

Here, C_Λ is the Λ-extension of C, an unbounded operator from X to Y, $D_1 \in \mathcal{L}(W, Y)$ and $D_2 \in \mathcal{L}(U, Y)$.

Two additional input signals to define well-posedness

In this case,

$$\mathbf{P}_1(s) = C_\Lambda(sI - A)^{-1}B_1 + D_1,$$

$$\mathbf{P}_2(s) = C_\Lambda(sI - A)^{-1}B_2 + D_2.$$

If Σ_p is not regular, then (2) has to be replaced by a more complicated formula, as discussed in the background on well-posed systems. A similar description applies to Σ_c.

For technical reasons, we consider two additional artificial input signals injected into the feedback system in Figure 1, $u_{\rm in}$ and $z_{\rm in}$, which are added to u and z, as shown in Figure 2. These signals could be interpreted as additional disturbances, noise, measurement errors or quantization errors. We consider the output signals to be $z_{\rm out}$ and $u_{\rm out}$, which are the output signals of Σ_p and of Σ_c.

Block diagram with two more inputs - Figure 2

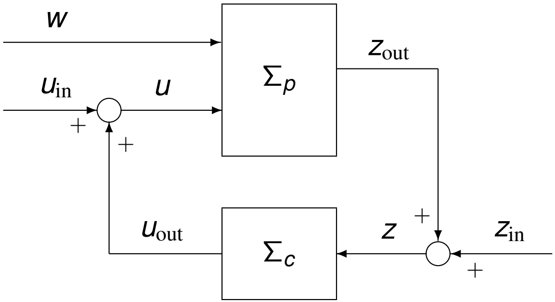

Figure: The closed-loop system from Figure 1, with two additional input signals u_{in} and z_{in}, which could be interpreted as noise or as measurement errors. From the three inputs to the two outputs, we have six relevant transfer functions. These, arranged in a 2×3 matrix, form the transfer function of the closed-loop system.

The exogenous signal

The controller Σ_c will be chosen such that the closed-loop system in Figure 2 (with inputs w, $u_{\rm in}$ and $z_{\rm in}$ and outputs $z_{\rm out}$ and $u_{\rm out}$) is well-posed and exponentially stable. In this case, the closed-loop system is described by equations similar to (1) and (2), and its state space is the Cartesian product of the state spaces of Σ_p and Σ_c.

Let $\mathcal{J}$ be a finite index set of integers. We assume that w is of the form

$$w(t) = \sum_{j\in\mathcal{J}} \mathrm{w}_j e^{i\omega_j t}, \quad \mathrm{w}_j \in W, \quad \omega_j \in \mathbb{R}. \tag{3}$$

Thus, w is a superposition of constant and sinusoidal signals. The frequencies ω_j are assumed to be known (for design purposes), but the vectors w_j (which determine the amplitudes and the phases) are not known in advance.

Some terminology and notation

For any $a \in \mathbb{R}$, we put

$$\mathbb{C}_a := \{s \in \mathbb{C} \,|\mathrm{Re}\, s > a\}.$$

For any Banach space Z, we denote by $H_a^\infty(Z)$ the space of bounded analytic Z-valued functions on $\mathbb{C}_a$. For $a = 0$ we also use the notation $H^\infty(Z)$. In the sequel, when the space Z is clear from the context, we write H_a^∞ for $H_a^\infty(Z)$ and H^∞ for $H^\infty(Z)$. A transfer function is called *stable* if it is in H^∞, and *exponentially stable* if it is in H_a^∞ for some $a < 0$. If a well-posed system is exponentially stable, then its transfer function is also exponentially stable. In particular, for the system in Figure 1, there exists $a < 0$ such that

$$\mathbf{P}_1 \in H_a^\infty(\mathcal{L}(W, Y)), \qquad \mathbf{P}_2 \in H_a^\infty(\mathcal{L}(U, Y)).$$

More notation, and objective of the paper under discussion

For any $\alpha \in \mathbb{R}$ we denote

$$L^2_\alpha[0,\infty) = \left\{ f \in L^2_{\rm loc}[0,\infty) \;\middle|\; \int_0^\infty e^{-2\alpha t}|f(t)|^2 {\rm d}t < \infty \right\}.$$

The corresponding space of Y-valued functions is denoted by $L^2_\alpha([0,\infty), Y)$ (or also $L^2([0,\infty), Y)$ if $\alpha = 0$). However, by some abuse of notation, we sometimes just write $L^2_\alpha[0,\infty)$ when the range space Y is clear from the context.

The objective of the paper is to find a controller Σ_c with transfer function **C** so that the closed-loop system in Figure 1 is exponentially stable, and the output z (the tracking error) decays exponentially to zero, by which we mean that $z \in L^2_\alpha[0,\infty)$ for some $\alpha < 0$.

The result of Hämäläinen and Pohjolainen

We now recall the main result of Hämäläinen and Pohjolainen (IEEE-TAC, 2000, HP). Their block diagram is slightly different, it corresponds to taking $\mathbf{P}_1 = [-I \;\; \mathbf{P}_2]$ in Figure 1, and they consider U and Y to be finite-dimensional, but these are not essential restrictions. They consider $\mathbf{P}$ to be an exponentially stable transfer function in the Callier-Desoer algebra. The internal model principle of Davison, Wonham and Francis suggests that $\mathbf{C}$ should have poles at $\{i\omega_j \mid j \in \mathcal{J}\}$. Following this principle, the following controller transfer function has been proposed and analyzed in HP:

$$\mathbf{C}(s) = -\varepsilon \sum_{j \in \mathcal{J}} \frac{K_j}{s - i\omega_j}, \qquad (4)$$

with

$$K_j \in \mathcal{L}(Y, U), \qquad \sigma(\mathbf{P}_2(i\omega_j)K_j) \subset \mathbb{C}_0 . \qquad (5)$$

Comments on the result of HP

Note that (5) implies that $\mathbf{P}_2(i\omega_j)K_j$ is invertible, hence the range of $\mathbf{P}_2(i\omega_j)$ is Y (i.e., the matrix $\mathbf{P}_2(i\omega)$ is onto at the relevant frequencies). It was shown in HP that for all $\varepsilon > 0$ sufficiently small, the feedback system in Figure 1 (with the assumptions of HP just described) has exponentially stable transfer functions and moreover, if w is as in (3), then the error z tends to zero. The approach of HP is algebraic and they consider also multiple poles in $\mathbf{C}$, which are needed if we want to allow the coefficients w_j in (3) to be polynomials in t (we have not reproduced the formulas corresponding to the multiple poles, since we only consider constant w_j).

This result is important because it allows tracking and disturbance rejection for external signals as in (3) *with very little information about the plant*. Indeed, all we have to know is that the plant is stable and we need some (possibly not precise) estimate of $\mathbf{P}_2(i\omega_j)$.

The first main theorem

Theorem 1. Suppose that Σ_p is an exponentially stable well-posed linear system with transfer function $\mathbf{P} = [\mathbf{P}_1 \ \mathbf{P}_2]$, where $\mathbf{P}_1(s) \in \mathcal{L}(W, Y)$, $\mathbf{P}_2(s) \in \mathcal{L}(U, Y)$. Let Σ_c be an exactly controllable and exactly observable realization of a transfer function $\mathbf{C}$ of the form

$$\mathbf{C}(s) = -\varepsilon \left(\mathbf{C}_0(s) + \sum_{j \in \mathcal{J}} \frac{K_j}{s - i\omega_j} \right), \tag{6}$$

where $\mathbf{C}_0 \in H^\infty_\alpha(\mathcal{L}(Y, U))$ with $\alpha < 0$, $K_j \in \mathcal{L}(Y, U)$ and $\sigma(\mathbf{P}_2(i\omega_j)K_j) \subset \mathbb{C}_0$.

Then the closed-loop system shown in Figure 2 is well-posed and exponentially stable for all sufficiently small $\varepsilon > 0$. For any such ε there exists $\beta < 0$ such that, in the system of Figure 1, if w is of the form (3), then $z \in L^2_\beta[0, \infty)$.

Comments on the first main theorem

The smaller ε, the smaller $|\beta|$, hence we would like large ε.

Note that, in (6), we have added the extra term $\mathbf{C}_0$ (when compared to (4)). The theorem would of course remain valid without this extra term, but $\mathbf{C}_0$ may be needed to satisfy some other design requirements, possibly derived from robustness considerations, or to shorten the transient response.

This theorem has not been stated in the strongest possible form: it is a consequence of a more general theorem where the conditions “exactly controllable” and “exactly observable” are replaced with the less restrictive “optimizable” and “estimatable”. If U and Y are finite-dimensional and $\mathbf{C}$ is rational (as it would be in most applications), then it is natural to take Σ_c to be a minimal realization of $\mathbf{C}$, so that Σ_c would be controllable and observable, as required in the theorem.

How about real Hilbert spaces?

In most physically motivated applications, W is the complexification of a real Hilbert space W_0, so that any $\mathrm{w} \in W$ has a unique decomposition $\mathrm{w} = \mathrm{w}^0 + i\mathrm{w}^1$ with $\mathrm{w}^0, \mathrm{w}^1 \in W_0$ and so the complex conjugate $\overline{\mathrm{w}} = \mathrm{w}^0 - i\mathrm{w}^1$ is well defined. Moreover, w takes values in W_0, which (with proper indexing) implies that

$$\omega_{-j} = -\omega_j, \quad \text{and} \quad \mathrm{w}_{-j} = \overline{\mathrm{w}}_j .$$

Similarly, U and Y are usually the complexifications of real Hilbert spaces U_0 and Y_0. **P** is *real*, i.e.,

$$\mathbf{P}_1(-i\omega)\mathrm{w}_0 = \overline{\mathbf{P}_1(i\omega)\mathrm{w}_0} \qquad \forall\, \mathrm{w}_0 \in W_0 ,$$

and a similar condition is satisfied by $\mathbf{P}_2$. In this case, **C** can be chosen to be real as well, by choosing

$$K_{-j}\mathrm{y}_0 = \overline{K_j\mathrm{y}_0} \qquad \forall\, \mathrm{y}_0 \in Y_0 ,$$

and by choosing $\mathbf{C}_0$ to be real.

Positive transfer functions

Let **P** be an $\mathcal{L}(U)$-valued transfer function defined on (a set containing) the half-plane $\mathbb{C}_0$. We say that **P** is a *positive* transfer function if

$$\mathrm{Re}\,\mathbf{P}(s) := \frac{1}{2}\left[\mathbf{P}(s) + \mathbf{P}(s)^*\right] \geq 0 \qquad \forall\, s \in \mathbb{C}_0 .$$

If a well-posed system is *impedance passive*, meaning that for some positive operator $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \Pi x(t), x(t)\rangle \leq 2\mathrm{Re}\,\langle u(t), y(t)\rangle ,$$

then the transfer function of this system is positive.

When the second component of the plant transfer function (from control input to error) is positive, the following theorem states that certain simple controllers will stabilize the system in the sense of Theorem 1 and also achieve tracking. Moreover, in this situation, there is no need to adjust an unknown small gain.

The second main theorem

Theorem 2. Suppose that Σ_p is an exponentially stable well-posed linear system with transfer function $\mathbf{P} = [\mathbf{P}_1 \ \mathbf{P}_2]$, where $\mathbf{P}_1(s) \in \mathcal{L}(W, U)$, $\mathbf{P}_2(s) \in \mathcal{L}(U)$, $\mathbf{P}_2$ is a positive transfer function, and $\mathrm{Re}\, \mathbf{P}_2(i\omega_j)$ is invertible for all $j \in \mathcal{J}$. Let Σ_c be an exactly controllable and exactly observable realization of a transfer function $\mathbf{C}$ of the form

$$\mathbf{C}(s) = -\left(\mathbf{C}_0(s) + \sum_{j\in\mathcal{J}} \frac{K_j}{s - i\omega_j}\right), \qquad (7)$$

where $K_j \in \mathcal{L}(U)$, $K_j > 0$, $K_j^{-1} \in \mathcal{L}(U)$, $\mathbf{C}_0 \in H^\infty_\alpha(\mathcal{L}(U))$ with $\alpha < 0$ and

$$\mathrm{Re}\, \mathbf{C}_0(s) \geq \frac{1}{2} I \qquad \forall\, s \in \mathbb{C}_0 .$$

Then the feedback system in Figure 2 is well-posed and exponentially stable. Moreover if w is of the form (3), then $z \in L^2_\beta[0, \infty)$ for some $\beta < 0$.

Background: optimizable systems

Definition. Let Σ be a well-posed linear system with semigroup generator A and control operator B. The system Σ (or the pair (A, B)) is called *optimizable* if for every $x_0 \in X$ there exists a $u \in L^2([0, \infty); U)$ such that $x \in L^2([0, \infty); X)$, where x is the state trajectory defined by $x(t) = \mathbb{T}_t x_0 + \Phi_t u$.

The system Σ (or the pair (A, B)) is called *exactly controllable* if for some $t > 0$, Φ_t is onto.

Optimizability is the most natural extension of the the concept of stabilizability from finite-dimensional systems to the context of well-posed systems. Motivated by linear quadratic optimal control theory, this property is sometimes called *the finite cost condition*. It is easy to see that exact controllability implies optimizability.

Background: estimatable systems

Definition. Let Σ be a well-posed linear system with semigroup generator A and observation operator C. The system Σ (or the pair (A, C)) is *estimatable* if (A^*, C^*) is optimizable. The system Σ (or the pair (A, C)) is *exactly observable* if (A^*, C^*) is exactly controllable.

Estimatability is equivalent to the solvability of a certain final state estimation problem, see a detailed paper about optimizability and estimatability by Weiss and Rebarber (SICON, 2000), denoted WR. Since it is the dual concept of optimizability, estimatability is an extension of the concept of detectability from finite-dimensional systems to the context of well-posed systems. Exact observability is equivalent to the fact that there exist $T > 0$ and $K_T > 0$ such that

$$\int_0^T \|C\mathbb{T}_t x_0\|^2 \mathrm{d}t \geq K_T \|x_0\|^2 \qquad \forall\, x_0 \in \mathcal{D}(A).$$

Via duality, exact observability implies estimatability.

Exponential stability via input-output stability

It is easy to see that if a well-posed system is exponentially stable, then it is optimizable and estimatable. Optimizability and estimatability are invariant under static output feedback (this is also easy to see - try it as an exercise).

A well-posed system is called *input-output stable* if its transfer function is in H^∞, i.e., it is bounded on $\mathbb{C}_0$. Equivalently, its extended input-output map transforms L^2 inputs into L^2 outputs. One of the main results of WR is the following:

Theorem 3. A well-posed system is exponentially stable if and only if it is optimizable, estimatable and input-output stable.

This has some interesting consequences. For instance, if a system is not exponentially stable but it is input-output stable, then there is no way to stabilize it exponentially using a well-posed linear controller.

A result on dynamic stabilization

The following theorem is a consequence of the last theorem.

Theorem 4. Suppose that Σ_p and Σ_c are well-posed linear systems with transfer functions denoted $\mathbf{P}$ and $\mathbf{C}$ (where $\mathbf{P} = [\mathbf{P}_1 \ \ \mathbf{P}_2]$), connected in feedback as shown in Figure 2. Suppose that Σ_p is optimizable via u, Σ_c is optimizable, and both systems are estimatable. Suppose that the four transfer functions

$$(I - \mathbf{P}_2\mathbf{C})^{-1}, \ \mathbf{C}(I - \mathbf{P}_2\mathbf{C})^{-1}, \ (I - \mathbf{P}_2\mathbf{C})^{-1}\mathbf{P}_2, \ \mathbf{C}(I - \mathbf{P}_2\mathbf{C})^{-1}\mathbf{P}_2 \quad (8)$$

are all stable (i.e., in H^∞). Then the closed-loop system $\Sigma_{p,c}$ shown in Figure 2 is an exponentially stable well-posed linear system.

The proof of Theorem 4

Proof. Under the given optimizability and estimatability hypotheses, it follows from Theorem 3 and the feedback theory of well-posed systems that the closed-loop system $\Sigma_{p,c}$ is well-posed and exponentially stable if (and only if) $(I - \mathbf{L})^{-1} \in H^\infty(U \times Y)$, where

$$\mathbf{L} = \begin{bmatrix} 0 & \mathbf{C} \\ \mathbf{P}_2 & 0 \end{bmatrix}.$$

It is easy to verify that $(I - \mathbf{L})^{-1} \in H^\infty$ if (and only if) the transfer functions listed in (8) are all in H^∞. ■

The transfer functions appearing in (8) are closely related to the transfer functions of the system in Figure 2, with $w = 0$. For example, $(I - \mathbf{P}_2\mathbf{C})^{-1}$ is the transfer function from $z_{\rm in}$ to z, hence $(I - \mathbf{P}_2\mathbf{C})^{-1} - I$ maps $z_{\rm in}$ to $z_{\rm out}$.

A key lemma

To prove the stability parts of Theorems 1 and 2 we show that an appropriate set of closed-loop transfer functions is stable, and then we apply Theorem 4 to conclude that the closed-loop system is exponentially stable.

To be able to apply Theorem 4, a key step is to show that the first of the four transfer functions listed in (8), called the *sensitivity* and denoted by **S**, is stable (i.e., in H^∞) for all sufficiently small $\varepsilon > 0$.

Lemma 5. Suppose that $\mathbf{P}_2 \in H^\infty_a(\mathcal{L}(U, Y))$ with $a < 0$ and **C** is given by (6) and satisfies the conditions listed after (6). Then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$,

$$\mathbf{S} := (I - \mathbf{P}_2\mathbf{C})^{-1} \in H^\infty .$$

The end is near!

THANK YOU!